ON THE DIRECT PRODUCT OF W^*-ALGEBRAS

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Introduction. From the algebraical point of view, the direct product of operator algebras is useful for the study of operator algebras. This notion is originally due to F. J. Murray and J. von Neumann [9, 10]. They have investigated chiefly the direct product of a factor of type I and a factor in general type. Recently, T. Turumaru [17, 18] has investigated the direct product of C^*-algebras and obtained many interesting results.

The present paper is devoted to a natural step in the direct product of W^*-algebras. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be W^*-algebras on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. Then the direct product $\mathbf{M}_1 \otimes \mathbf{M}_2$ of $\mathbf{M}_1$ and $\mathbf{M}_2$ is defined as the weak closure of the algebraical direct product of $\mathbf{M}_1$ and $\mathbf{M}_2$ on the Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. Therefore the concept of the direct product of W^*-algebras depends on the underlying Hilbert spaces. One of purposes of this paper is to show that $\mathbf{M}_1 \otimes \mathbf{M}_2$ does not depend on underlying Hilbert spaces in algebraical sense. That is, if $\mathbf{M}_1$, $\mathbf{M}_2$ are represented as W^*-algebras on another Hilbert spaces $\mathfrak{K}_1$, $\mathfrak{K}_2$ respectively, then $\mathbf{M}_1 \otimes \mathbf{M}_2$ on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is algebraically $*$-isomorphic (in the following we shall state isomorphic) to the one on $\mathfrak{K}_1 \otimes \mathfrak{K}_2$ (Theorem 1).

The study of the relation between the semi-finite W^*-algebras and the Hilbert algebras is fostered by F. J. Murray and J. von Neumann and developed by J. Dixmier, H. A. Dye, R. Godement, I. E. Segal and many authors. R. Pallu de la Barrière has investigated the direct product of Hilbert algebras. Our second purpose is to study of the relations between the direct product of Hilbert algebras and the one of W^*-algebras which is stated in Theorem 2. This result is the central role for the study of the direct product of semi-finite W^*-algebras. For example, as an application of it, we shall prove the commutation theorem for the direct product in semi-finite case (Theorem 3).

The final section is devoted to the direct product of finite W^*-algebras. We shall show that the direct product of finite W^*-algebras is finite too (Theorem 4) and further we shall consider the type of direct product in semi-finite case. There is a $\natural$-operation in a finite W^*-algebra in the sense of J. Dixmier [2], and we shall consider the relation of $\natural$-operations between the direct product of finite W^*-algebras and them. An approximately finite factor is a factor of type II_1, which has simple construction, and seems fundamental for the study of factors of type II_1. We shall consider the direct product of these factors and prove that this product is approximately finite (Theorem 6). The last theorem states that the fundamental group of the direct product of two finite factors contains the fundamental group of each factors as its subgroup.

1. The direct product of W^*-algebras. In this section, we shall give