INDUCTIVE LIMIT AND INFINITE DIRECT PRODUCT OF OPERATOR ALGEBRAS

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Introduction. The first hurdle for the definition of the direct product of operator algebras was the construction of the product space on which the product algebra acts as an operator algebra. This task was accomplished by F. J. Murray and J. von Neumann in the prominent papers [6] [8]. However by the recent progress of the theory of operator algebras, for some problems we can now proceed without the explicit usage of underlying Hilbert spaces. Thus T. Turumaru [14] has succeeded to define the direct product of two C^*-algebras without regard to the product space and Y. Misonou [5] [13] [14] has proved that the algebraical structure of the direct product of two W^*-algebras is uniquely determined only by the component algebras themselves and is independent from Hilbert spaces they act. The purpose of this paper is to extend these results to infinite direct product of operator algebras.

We introduce in § 1 a notion of inductive limit of C^*-algebras and get a dual relation: the state space of an inductive limit of C^*-algebras A_γ ($\gamma \in \Gamma$) is the projective limit of state spaces of A_γ ($\gamma \in \Gamma$). After a brief résumé of the finite direct product in § 2, using the concept of inductive limit we define the infinite direct product of C^*-algebras in § 3. As seen in the construction of the infinite product of measure spaces [3], this infinite product has been already popular for commutative algebras in some implicit forms. For this product the associative law holds in the satisfactory manner.

Next, in the recent literatures [1] [2] [9], it is known that the σ-weak topology[1] plays a striking rôle in the theory of W^*-algebras and a state is σ-weakly continuous if and only if it is normal[2] in the sense of Dixmier. Hence employing normal states instead of ordinary states and regarding the dual relation stated above we define the inductive limit of W^*-algebras in §4. We take a W^*-algebra A as the inductive limit of W^*-algebras A_γ ($\gamma \in \Gamma$) if the set of normal states of A coincides with the projective limit of normal states on A_γ ($\gamma \in \Gamma$). Then this W^*-inductive limit is uniquely determined. In § 5, relaxing the condition of W^*-inductive limit, we consider a limit of W^*-algebras on a fixed Hilbert space which is named a direct limit. An approximately finite factor [7] gives an example of the direct limit. The relation between the inductive limit and the direct limit is given by the following fact: a direct limit of W^*-algebras A_γ ($\gamma \in \Gamma$) is always a normal homomorphic image of the W^*-inductive limit of A_γ ($\gamma \in \Gamma$). In § 6, parallel to the direct product of C^*-algebras we define an infinite direct product of W^*-algebra.

1) This terminology is due to Griffin [2]. Dixmier [1] called it *la topologie ultrafaible*.

2) A state σ is called normal [1] if $x_\alpha \uparrow x$ implies $\sigma(x_\alpha) \uparrow \sigma(x)$.