ON ASYMPTOTIC CONJUGATE POINTS

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In a simply connected Riemannian space with non-positive curvature, E. Cartan [1] dealt with a particular class of geodesics called asymptotes. H. Busemann [2] extended the concept of asymptotes to the case of E-spaces defined by him. In this note the initial point of an asymptote will be called, if it exists, its asymptotic conjugate point (§2). We study the set of asymtotic conjugate points to a given ray and show some examples to make clear the circumstances.

1. In this paragraph we explain some preliminary concepts.

In a metric space[1)] points will always be denoted by small roman letters and the distance between two points x and y will be denoted by xy. The axioms for a space $\mathfrak{E}$ to be an E-space are these:

A. $\mathfrak{E}$ is metric with distance xy not necessarily symmetric.

B. $\mathfrak{E}$ is finitely compact, i. e., every bounded subset X contains a sequence of points $\{x_\nu\}$ which converges to a point x in $\mathfrak{E}$.

C. $\mathfrak{E}$ is convex metric, i. e., for every pair of two distinct points x and y a point z with $xz + zy = xy$ exists.

D. Every point x has a spherical neighborhood $s(x, \alpha_x) = \{y \mid yx < \alpha_x,\ xy < \alpha_x\}(\alpha_x > 0)$ such that for any positive number ε and any two points $a, b \in S(x, \alpha_x)$ there exists positive numbers $\delta_k(a, b)\ (\leqq \varepsilon)\ (k = 1, 2)$ for which a point a_1 with $a_1a + ab = a_1b$ and $a_1a = \delta_1$ and another point b_1 with $ab + bb_1 = ab_1$ and $bb_1 = \delta_2$ exist and are unique.

In an E-space, for any two points x and y the axioms A, B, and C guarantee the existence of a segment $T(x, y)$ (or $T(y, x)$) from x to y (or from y to x) whose length is equal to the distance xy (or yx). Furthermore the prolongation of an arbitrary segment is locally possible and unique under the axiom D. The whole prolongation of a segment is called an extremal. An extremal $\mathfrak{x}$ has a parametric representation $x(\tau)$, $-\infty < \tau < +\infty$, such that for every τ_0 a positive number $\delta(\tau_0)$ exists such that $x(\tau_1)\,x(\tau_2) = \tau_2 - \tau_1$ for $\tau_2 \geqq \tau_1$ and $|\tau_0 - \tau_i| \leqq \delta(\tau_0)\ (i = 1, 2)$.

An extremal $\mathfrak{x}$ is called a straight line, if its parametric representation $x(\tau)$, $-\infty < \tau < +\infty$, has the property $x(\tau_1)x(\tau_2) = \tau_2 - \tau_1(\tau_2 \geqq \tau_1)$. A positive ray is a positive half extremal $x(\tau)$, $0 \leqq \tau < +\infty$, for which $x(\tau_1)x(\tau_2) = \tau_2 - \tau_1(\tau_2 \geqq \tau_1)$. Similarly negative rays are defined. Our consideration will be

1) In this paper, we use the "metric space" for a set $\mathfrak{E}$ with the following properties (1), (2), (3), (4).

(1) xy is defined for any ordered pair of points x and y in $\mathfrak{E}$ and is non-negative.

(2) $xy=0$, if and only if $x=y$.

(3) $xy+yz \geqq xz$ for any three points x, y, and z.

(4) For any sequence of points $\{x_\nu\}$ and a point x, $xx_\nu \to 0$ if and only if $x_\nu x \to 0$.