ALMOST HERMITIAN STRUCTURE ON S^6

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A differentiable manifold M of even dimension is called to have an *almost Hermitian structure*, if there exist on M a metric tensor field g and a tensor field φ of type (1, 1) such that

$$\varphi^i_a \varphi^a_j = -\delta^i_j, \qquad g_{ab}\varphi^a_i \varphi^b_j = g_{ij}.$$

It is well known that an almost Hermitian structure can be defined on a sphere S^6 of dimension 6 by making use of the algebra C of Cayley numbers [1]. The group G of all automorphisms of the algebra C acts on S^6 transitively as a group of isometries [3]. We shall show that almost Hermitian structure on S^6 is invariant under the group G, i. e. the tensor fields g and φ are invariant under G. Moreover there exists one and only one affine connection Λ invariant under G with respect to which g and φ are covariantly constant. The unique connection Λ is actually obtained by making use of the fields g and φ, and its torsion and curvature tensor fields are covariantly constant.

1. Almost Hermitian structure on S^6. Let C be the algebra of Cayley numbers. Any element ξ of C may be written in the form

$$\xi = XI + X^0 e_0 + X^1 e_1 + \dots + X^6 e_6 = XI + x,$$

where $X, X^0, X^1, \dots, X^6$ are real numbers and $I, e_0, e_1, \dots, e_6$ form a natural base of the algebra C, I being the unit element of C. ξ is called pure imaginary if $X = 0$. All pure imaginary Cayley numbers form a 7-dimensional subspace E^7 of C. The multiplication table is given by the following:

$$\begin{array}{llll} e_i^2 = -I & & (i = 0, 1, \dots, 6), & \\ e_i \cdot e_j = -e_j \cdot e_i & & (i \neq j;\ i, j = 0, 1, \dots, 6), & \\ e_0 \cdot e_1 = e_2, & e_0 \cdot e_3 = e_4, & e_0 \cdot e_5 = e_6, & \\ e_1 \cdot e_4 = e_6, & e_1 \cdot e_3 = -e_5, & e_2 \cdot e_3 = e_6, & e_2 \cdot e_4 = e_5, \end{array}$$

and the other $e_i \cdot e_j$ are given by cyclic permutations of indices.

If $\eta = YI + \sum_{=0}^{6} Y^i e_i = YI + y$ is an element of C, then we have

$$\xi \cdot \eta = XYI - \sum_{i=0}^{6} X^i Y^i I + Xy + Yx + \sum_{i,k=0}^{6}{}_{i \neq k} X^i Y^k e_i \cdot e_k.$$

For any two pure imaginary Cayley numbers x, y we have

$$x \cdot y = -\sum_{i=0}^{6} X^i Y^i I + \sum_{i,k=0}^{6}{}_{i \neq k} X^i Y^k e_i \cdot e_k,$$

where $\sum_{i=0}^{3} X^i Y^i$ is the *scalar product* of two vectors x and y of E^7 and is denoted by (x, y). The pure imaginary Cayley number $\sum_{i \neq k} X^i Y^k e_i e_k$ is called