ON ENTIRE FUNCTIONS OF INFINITE ORDER

QAZI IBADUR RAHMAN

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1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of infinite order and let $M(r)$, $M'(r)$, $M(r, f^{(p)})$, $\mu(r)$, $\nu(r) \equiv \nu(r, f)$ and $\nu(r, f^{(p)})$ have their usual meanings. It is known [3, (80-1)] that

$$\lim_{r \to \infty} \frac{\log \mu(r)}{\nu(r)} = 0. \qquad (1.\ 1)$$

A result better than (1. 1) viz.,

$$\lim_{r \to \infty} \frac{\log M(r)}{\nu(r)} = 0, \qquad (1.\ 2)$$

for every entire function of infinite order has been proved by Shah [4, (113-4)]. Later on Shah and Khanna [5, (47-8)] proved that for an entire function of infinite order

$$\lim_{r \to \infty} \frac{\log \{rM'(r)\}}{\nu(r, f)} = 0, \qquad (1.\ 3)$$

—— a result better than (1. 2) since [6, (116)]

$$rM'(r) > M(r) \frac{\log M(r)}{\log r}, \quad r \geqq r_0 = r_0(f).$$

Clunie [1] has gone still further to prove that if s is any function of ν such that $s(\nu) = o\left(\frac{\nu}{\log \nu}\right)$, then

$$\lim_{r \to \infty} \frac{\log \{r^s M(r, f^{(s)})\}}{\nu(r, f)} = 0. \qquad (1.\ 4)$$

In § 2 of this note I give an alternative proof of (1. 4). In § 3 I prove still another result better than (1. 1).

2. We have

$$r^s M(r, f^{(s)}) \leqq \sum_{n=s}^{\infty} n(n-1)\dots(n-s+1)\,|a_n|\,r^n$$

in the notation of G.Valiron, [7, (30)] for $n \geqq p$

$$\begin{aligned} n(n-1)\dots(n-s+1)\,|a_n|\,r^n &\leqq n(n-1)\dots(n-s+1)e^{-G_n} r^n \\ &\leqq n(n-1)\dots(n-s+1)\mu(r)\left(\frac{r}{R_p}\right)^{n-p+1} \end{aligned}$$