ON THE JACOBSON RADICAL OF A SEMIRING

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The concept of the Jacobson radical of a semiring has been introduced internally by S. Bourne ([1]). Recently, by associating a suitable ring with the semiring, S. Bourne and H. Zassenhaus have defined the semiradical of the semiring ([2]). In [3] it has been proved that the concepts of the Jacobson radical and the semiradical coincide. Consequently some properties of the Jacobson radical of the semiring are reduced to those in the ring theory. For example, the fact "If R is the Jacobson radical of a semiring S with a unit element, then R_n is the Jacobson radical of the matrix semiring S_n ([1])" is deduced immediately from the corresponding result in the ring theory.

The purpose of this paper is to consider the Jacobson radical of a semiring from the point of view of the representation theory[1] without reducing it to the ring theory. In § 1 we shall describe some preliminary definitions and propositions. In § 2, we shall define the irreducible representations and the radical of a semiring and prove some fundamental properties of the radical which correspond to those in the ring theory. In § 3, the external notion of the radical will be related to internal one, at the same time, we shall see that the radical defined in this paper coincides with the Jacobson radical and with the semiradical of the semiring. In the last section we shall consider some of the results obtained in the preceding sections from the point of view of the ring theory and give some examples.

1. Preliminaries. In this paper we shall assume that a *semiring* $\mathfrak{A}$ is commutative relative to addition and has a zero element. A commutative additive semigroup $\mathfrak{M}$ with a zero element is called a *right* $\mathfrak{A}$ *semimodule* if and only if a law of composition on $\mathfrak{M} \times \mathfrak{A}$ into $\mathfrak{M}$ is defined which, for $x, y \in \mathfrak{M}$ and $a, b \in \mathfrak{A}$, satisfies (a) $(x+y)a = xa + ya$, (b) $x(a+b) = xa + xb$ and (c) $x(ab) = (xa)b$. Henceforth the term "$\mathfrak{A}$-semimodule" without modifier will always mean right $\mathfrak{A}$-semimodule. The semiring $\mathfrak{A}$ itself is an $\mathfrak{A}$-semimodule relative to right multiplication as semimodule composition. A subset $\mathfrak{N}$ of $\mathfrak{M}$ is called an $\mathfrak{A}$-*subsemimodule* of $\mathfrak{M}$ if and only if

1) This notion has been used by N. Jacobson ([5]).