SOME RESULTS ON THE DIRECT PRODUCT OF W^*-ALGEBRAS

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Introduction. In connection with the papers [4] and [5], the following question arises: *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be finite factors, and let* G *and* H *be the groups of* $*$*-automorphisms of* $\mathbf{M}$ *and* $\mathbf{N}$ *respectively. Then, is it true that the fixed algebra of* $G \times H$[1] *in* $\mathbf{M} \otimes \mathbf{N}$ *is the direct product of the fixed algebra of* G *in* $\mathbf{M}$ *and that of* H *in* $\mathbf{N}$? The above question motivates the preparation of this paper, but our investigation will be done from the standpoint of the general theory of the direct product of W^*-algebras, and the main result will be stated in Theorem 1 in § 1. In § 2, as the applications of Theorem 1, two results will be proved. Theorem 2 is a structure theorem on the direct product of maximal abelian W^*-subalgebras, and Theorem 3 gives the affirmative answer to the question of the fixed algebra.

1. Throughout our discussion, we mean by $R(A_\alpha)$ the W^*-algebra generated by the family of operators A_α and $R(\mathbf{M}, \mathbf{N})$ the one generated by the W^*-algebras $\mathbf{M}, \mathbf{N}$.

The following theorem is the main result of this paper.

THEOREM 1. *Let* $\mathbf{M}, \mathbf{P}$ *and* $\mathbf{N}, \mathbf{Q}$ *be* W^**-algebras on some Hilbert space* $\mathbf{H}$ *and* $\mathbf{K}$ *respectively, and satisfy the condition*

(1) $\quad ((\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}'))' = (\mathbf{M} \cap \mathbf{P}')' \otimes (\mathbf{N} \cap \mathbf{Q}')'$;

then we have

(2) $\quad \mathbf{M} \otimes \mathbf{N} \cap (\mathbf{P} \otimes \mathbf{Q})' = (\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}')$.

This theorem shows that if the, so-called, commutation theorem holds for a W^*-algebra $(\mathbf{M} \cap \mathbf{P}') \otimes (\mathbf{N} \cap \mathbf{Q}')$ we get the conclusion (2). Hence, for example, if $\mathbf{A}$ and $\mathbf{B}$ are maximal abelian W^*-subalgebras of $\mathbf{M}$ and $\mathbf{N}$ respectively we have the relation (2) for $\mathbf{A} \otimes \mathbf{B}$ because, in this case, $((\mathbf{M} \cap \mathbf{A}') \otimes (\mathbf{N} \cap \mathbf{B}'))' = (\mathbf{A} \otimes \mathbf{B})' = \mathbf{A}' \otimes \mathbf{B}'$. Therefore we know that the direct product of maximal abelian W^*-subalgebras $\mathbf{A}$ and $\mathbf{B}$ is also a maximal abelian W^*-subalgebra of $\mathbf{M} \otimes \mathbf{N}$. If $\mathbf{M}$ and $\mathbf{N}$ are finite W^*-algebras, their W^*-subalgebras are also of finite type, and hence the commutation theorem always holds for these W^*-algebras. Therefore we have the conclusion

1) For the definition of $G \times H$, see Lemma 2 in [4].