ON THE CLASS OF SATURATION IN THE THEORY OF APPROXIMATION II[1)]

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1. Introduction. We have already determined the class of saturation for various methods of summation in the theory of Fourier series and Fourier integral (Sunouchi-Watari [4], Sunouchi [5]). In the present note, the author will determine the saturation class of local approximation of functions. In the preceding case, we have used Fourier transform or Fourier series essentially. But in the present case, we have to make another device. For the sake of simplicity, we will only study about the uniform approximation by the first arithmetic means. But it is evident that our method is applicable for another approximation norms and another summation methods.

Incidently, this method is applicable to determine the saturation class for such approximation process as Landau's singular integral. This approximation process has somewhat different feature from the periodic case.

2. Local saturation. Let $f(x)$ be integrable and periodic with period 2π and $\sigma_n(x, f) \equiv \sigma_n(x)$ be the n-th Fejér means of the Fourier series of $f(x)$. Moreover we suppose $[a, b]$ is a fixed closed subinterval situated in $[0, 2\pi]$. Then we have the following theorem.

THEOREM 1[2)]. (1) *If* $\sigma_n(x) - f(x) = o(1/n)$ *uniformly in* $[a, b]$, *then* $\tilde{f}(x)$ *is a constant in* $[a, b]$.

(2) *If* $\sigma_n(x) - f(x) = O(1/n)$ *uniformly in* $[a, b]$, *then* $\tilde{f}'(x)$ *is essentially bounded in* $[a, b]$.

For the proof of Theorem 1, we need a lemma.

LEMMA. *If* $h(x)$ *is periodic with period* 2π *and* $\tilde{h}''(x)$ *is continuous in* $[0, 2\pi]$, *then*

$$\lim_{n\to\infty} n\{\sigma_n(x, h) - h(x)\} = \tilde{h}'(x)$$

bourdedly.

PROOF. Interchanging the roles of $h(x)$ and the conjugate of $h(x)$, we shall

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2) The author assumed previously that $\tilde{f}(x)$ is integrable. He is obliged to Professor Zygmund for removing this assumption.