A GAP SEQUENCE WITH GAPS BIGGER THAN THE HADAMARD'S

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1. Introduction. In the present note let $f(t)$ be a measurable function satisfying the conditions ;

$$(1.1) \qquad f(t+1)=f(t), \int_0^1 f(t)dt=0 \text{ and } \int_0^1 f^2(t)dt=1.$$

In [2] M. Kac noticed that if $f(t)$ is a function of Lip α or of bounded variation, then it is seen that

$$(1.2) \qquad \lim_{N\to\infty}\left|\left\{t;\frac{1}{A_N}\sum_{k=1}^{N}a_kf(n_kt)<\omega\right\}\right|=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega}e^{-u^2/2}\,du,$$

where $\{n_k\}$ is a sequence of integers such that

$$(1.3) \qquad \lim_{k\to\infty} n_{k+1}/n_k=+\infty$$

and $\{a_k\}$ is any sequence of real numbers satisfying the following conditions

$$(1.4) \qquad A_N^2=\sum_{k=1}^{N}a_k^2\to+\infty \text{ and } \max_{1\leqq k\leqq N}|a_k|=o(A_N), \qquad \text{as } N\to+\infty.$$

Also in [4] G. Morgenthaler proved that if $f(t)$ is bounded and $\{a_k\}$ satisfies (1.4), then there exists a sequence $\{f(n_kt)\}$ independent of $\{a_k\}$ and (1.2) holds.

On the other hand P. Erdös [2] showed that if $f(t)=\cos 2\pi t+\cos 4\pi t$, then we have

$$\lim_{N\to\infty}\left|\left\{t;\frac{1}{\sqrt{N}}\sum_{k=1}^{N}f((2^k-1)t)<\omega\right\}\right|=\frac{1}{\sqrt{\pi}}\int_0^1dx\int_{-\infty}^{\omega/2|\cos\pi x|}e^{-u^2/2}du.$$

From above facts we see that if (1.2) holds, the properties of n_{k+1}/n_k and the smoothness of $f(t)$ become subjects of considerations (cf. [3]). The purpose of this note is to prove the following

THEOREM. *Let $\{n_k\}$ and $\{a_k\}$ satisfy* (1.3) *and* (1.4) *respectively and for some $\varepsilon>0$,*

$$(1.5) \qquad \left[\int_0^1\{f(t)-S_n(t)\}^2dt\right]^{1/2}=O\left(\frac{1}{(\log n)^{1+\varepsilon}}\right), \qquad as\ n\to+\infty,$$