# ON DISTORTIONS IN CERTAIN QUASICONFORMAL MAPPINGS

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Let $w = f(z)$ be a quasiconformal mapping of $|z| < 1$ into the $w$-plane in the sense of Pfluger-Ahlfors, whose maximal dilatation is not greater than a finite constant $K(\geqq 1)$, then it will be simply referred to a $K$-$QC$ mapping in $|z| < 1$.

First, we formulate, in §1, a theorem producing Schwarz-Pfluger's theorem [5], next determine in §2 the range of a real number $\alpha$ such that there is no positive finite $\lim_{z\to 0} |f(z)|/|z|^{\alpha}$ for any $K$-$QC$ mapping $w = f(z)$ in $|z| <1$ satisfying $f(0) = 0$, and finally in §3, we establish, as applications, some distortion theorems supplementing completely our preceding results [3].

**1.** A.Pfluger [5] reported that for any $K$-$QC$ mapping $w = f(z)$ of $|z| < 1$ onto $|w| < 1$ with the limit $\lim_{z\to 0} |f(z) - f(0)|/|z|^{1/K} = c$, $c \leqq 1 - |f(0)|^2 \leqq 1$ holds and $c = 1$ arises when $w = f(z) = e^{i\phi}z|z|^{(1/K)-1}$.

Now, we prove the following theorem producing the above Pfluger's result, and state its corollary.

THEOREM 1. *Let $w = f(z)$ be a $K$-$QC$ mapping of $|z| < 1$ onto $|w| < 1$ such that $f(0) = 0$. If $\alpha \leqq 1/K$, then there holds*

$$\liminf_{z\to 0} |f(z)|/|z|^{\alpha} \leqq 1,$$

*where the equality holds only if $f(z) = e^{i\phi}|z|^{1/K}e^{i\arg z}$ with a real constant $\phi$.*

PROOF. Denote by $L(r)$ and $A(r)$ respectively the length and the area of the images of $|z| = r$ and $|z| < r$ under $w = f(z)$. Then we have for almost all $r \in (0, 1)$,

$$L(r) = \int_0^{2\pi} \left| \frac{\partial f(re^{i\theta})}{r\partial\theta} \right| r\, d\theta,$$

and for arbitrary $r, r'$ such that $0 < r < r' < 1$,

$$A(r') - A(r) \geqq \int_0^{2\pi} \int_r^{r'} J[f(re^{i\theta})] r d\theta dr,$$

where $J[f]$ means the Jacobian of $f$.

By using Schwarz's inequality and the well known formula $|\partial f(re^{i\theta})/r\partial\theta|$