ON THE CROSS-NORM OF THE DIRECT PRODUCT OF C^*-ALGEBRAS

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(Received December 5, 1963)

In [7] Turumaru introduced the notion of the direct product of C^*-algebras. Let $\mathfrak{A}_1 \odot \mathfrak{A}_2$ be the algebraic direct product of two C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Then $\mathfrak{A}_1 \odot \mathfrak{A}_2$ becomes a $*$-algebra under the natural algebraic operations. Turumaru's C^*-direct product $\mathfrak{A}_1 \widehat{\otimes}_\alpha \mathfrak{A}_2$ is given as the completion of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ under the norm given by

$$\left\| \sum_{i=1}^n a_{1,i} \otimes a_{2,i} \right\| = \sup \frac{\varphi_1 \otimes \varphi_2 \left[\left(\sum_{j=1}^m x_{1,j} \otimes x_{2,j} \right)^* \left(\sum_{i=1}^n a_{1,i} \otimes a_{2,i} \right)^* \left(\sum_{i=1}^n a_{1,i} \otimes a_{2,i} \right) \left(\sum_{j=1}^m x_{1,j} \otimes x_{2,j} \right) \right]^{1/2}}{\varphi_1 \otimes \varphi_2 \left[\left(\sum_{j=1}^m x_{1,j} \otimes x_{2,j} \right)^* \left(\sum_{j=1}^m x_{1,j} \otimes x_{2,j} \right) \right]^{1/2}}$$

$\sum_{i=1}^n a_{1,i} \otimes a_{2,i} \in \mathfrak{A}_1 \odot \mathfrak{A}_2$, where φ_1, φ_2 run over the set of all states of $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\sum_{j=1}^m x_{1,j} \otimes x_{2,j}$ runs over $\mathfrak{A}_1 \odot \mathfrak{A}_2$. Let us call this norm T-cross norm. Then T-cross norm has the following property: If π_1 and π_2 are representations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ to Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively, then the naturally defined product representation $\pi_1 \otimes \pi_2$ of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ to the product Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is continuous with respect to T-cross norm, so that $\pi_1 \otimes \pi_2$ can be extented to the representation of $\mathfrak{A}_1 \widehat{\otimes}_\alpha \mathfrak{A}_2$ which is also denoted by $\pi_1 \otimes \pi_2$. Besides, if π_1 and π_2 are faithful then $\pi_1 \otimes \pi_2$ becomes faithful. Hence T-cross norm is very natural norm of $\mathfrak{A}_1 \odot \mathfrak{A}_2$. But it is another matter whether or not T-cross norm is unique compatible norm of $\mathfrak{A}_1 \odot \mathfrak{A}_2$, where a norm β of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is called compatible to the algebraic structure of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ if the completion of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ by β becomes a C^*-algebra and $\|x_1 \otimes x_2\|_\beta \leqq \|x_1\| \cdot \|x_2\|$ for $x_1 \in \mathfrak{A}_1$ and $x_2 \in \mathfrak{A}_2$. In the present note we shall answer for this question that T-cross norm is smallest among the compatible norms and that T-cross norm is unique in $\mathfrak{A}_1 \odot \mathfrak{A}_2$ for C^*-algebra $\mathfrak{A}_1$ of certain class but it is not so in general. So we say that C^*-algebra $\mathfrak{A}_1$ has the property (T) if the following is true;

(T): For every C^*-algebra $\mathfrak{A}_2$ T-cross norm of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is the unique compatible norm.

LEMMA 1. *If π is a $*$-representation of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ to a Hilbert space $\mathfrak{H}$ which*