ON DERIVATIONS OF NILPOTENT LIE ALGEBRAS

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By E. Schenkman and N. Jacobson [1], it is shown that a nilpotent Lie algebra over a field of characteristic 0 always has a non-inner derivation. That is, denote by $\mathfrak{D}(N)$ and $\mathfrak{J}(N)$ the derivation algebra and the collection of all inner derivations of a nilpotent Lie algebra N respectively, then $\mathfrak{D}(N) \supsetneqq \mathfrak{J}(N)$. Clearly $\mathfrak{J}(N)$ is contained in the maximal nilpotent ideal $\mathfrak{N}$ of $\mathfrak{D}(N)$, hence $\mathfrak{J}(N)$ is contained in the radical $\mathfrak{R}$ of $\mathfrak{D}(N)$. In the present note we shall show that $\mathfrak{R}$ does not coincide with $\mathfrak{J}(N)$. That is, the Schenkman-Jacobson's result may be strengthened as follows:

THEOREM. *Let $\mathfrak{D}(N)$ be the derivation algebra of a nilpotent Lie algebra N over a field of characteristic* 0, *then the radical $\mathfrak{R}$ of $\mathfrak{D}(N)$ always contains a non-inner derivation.*

To prove the theorem we prepare the following lemma.

LEMMA. *Let N be a nilpotent Lie algebra whose center $Z(N)$ is contained in the derived subalgebra $[N, N]$, and let us suppose that for any ideal M with codimension* 1 *of N, the center $Z(M)$ of M is not contained in $[N, N]$. Then N belongs to either of following two types:*

Type (i). *N has an ideal M with codimension* 1 *and an element e such that*

$$e \notin M,\ [e, [N, N]] = 0 \quad and \quad [e, Z(M)] \subsetneqq Z(N)\,.$$

Type (ii). *N has a basis $\{e_{11}, e_{12}, e_{21}, e_{22}, \cdots, e_{r1}, e_{r2}, z_0\}$ which satisfies the conditions*

$$[e_{ij}, e_{kl}] = 0 \qquad for \qquad i \neq k\,,$$

$$[e_{i1}, e_{i2}] = z_0\,,$$

$$[e_{ij}, z_0] = 0\,.$$

PROOF. Let M be an arbitrary ideal with codimension 1. We note first that such an ideal M always contains the derived subalgebra $[N, N]$. In fact,