Tôhoku Math. Journ.
20 (1968), 417-424.

GROWTH CONDITIONS AND THE NUMERICAL RANGE IN A BANACH ALGEBRA

J. G. STAMPFLI AND J. P. WILLIAMS*)

(Received January 22, 1968)

0. Introduction. In this paper, we consider the numerical range in an arbitrary Banach algebra with identity, and study its relation to various growth conditions on the resolvent.

In §1, we list several facts about this generalized numerical range. Some of these are more or less well known for concrete algebras, but do not seem to have been formulated in their proper generality. In particular, we recognize that part of this section is implicit or explicit in Lumer [7]. Consequently, we have included proofs here only when a result is new (or requires a new technique), or when the proof represents a substantial simplification.

The main result of §2 is a Phragmén-Lindelöf theorem for quasi-nilpotent elements in a Banach algebra. This yields a sharper version of a similar theorem of Lumer and Phillips [8].

Finally, in §3 we apply the methods of §1 to study the numerical range in its usual setting, Hilbert space. The notion of essential numerical range appears naturally here, and this set is shown to be chracterized in the way one would expect by analogy with the essential spectrum.

1. Let $\mathcal{A}$ be a complex Banach algebra with unit, and let $\mathfrak{p} = \{f \in \mathcal{A}^* : f(1) = 1 = \|f\|\}$ be the set of positive linear functionals on $\mathcal{A}$. (These are called normalized states in [7].) For $x \in \mathcal{A}$, define the *numerical range* $W_0(x)$ as

$$W_0(x) = \{f(x) : f \in \mathfrak{p}\}$$

and the *numerical radius* $|W_0(x)|$ as

$$|W_0(x)| = \sup\{|z| : z \in W_0(x)\} .$$

The results of §1 are concerned with the relation between $W_0(x)$ and first

*) The first author gratefully acknowledges the support of the National Science Foundation.