Tôhoku Math. Journ.
23 (1971), 49-58.

HOMOGENEOUS HYPERSURFACES IN A SPHERE WITH THE TYPE NUMBER 2

RYOICHI TAKAGI

(Received July 22, 1970)

0. Introduction. There is a problem of giving a complete classification of homogeneous hypersurfaces M^n in a sphere S^{n+1} of dimension $n+1 (n \geqq 2)$. This problem can be naturally divided into three cases:

(i) The rank of the second fundamental form (which is called the type number) is not smaller than 3 at some point.
(ii) The type number is equal to 2 at some point.
(iii) The type number is equal to 0 or 1 at some point.

In the case (i), it is known by a theorem of Ryan [9] that the full isometry group of every homogeneous hypersurface M^n can be considered as a subgroup of the orthogonal group $O(n+2)$, in other words, M^n is an orbit of a suitable subgroup of $O(n+2)$. Hsiang and Lawson [5] gave a complete list of compact minimal hypersurfaces in S^{n+1} each of which is an orbit of a subgroup of $O(n+2)$.

The condition "minimal" is not essential because among all homogeneous hypersurfaces obtained as orbits of a compact subgroup of $O(n+2)$ there is a minimal one ([5]). Thus our problem is solved in the case (i) if the hypersurfaces are compact.

The purpose of this paper is to determine all hypersurfaces in S^{n+1} in the case (ii). To describe our results, we begin with an example of homogeneous hypersurface in S^4. Let $S^n(c)$ denote an n-dimensional sphere in Euclidean $(n+1)$-space R^{n+1} with curvature c. We consider the hypersurface in $S^4 = S^4(1)$ defined by the equations

$$(1) \qquad \begin{cases} 2x_2^3 + 3(x_1^2 + x_2^2)x_5 - 6(x_3^2 + x_4^2)x_5 + 3\sqrt{3}(x_1^2 - x_2^2)x_4 \\ \qquad\qquad + 3\sqrt{3}x_1 x_2 x_3 = 2, \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1. \end{cases}$$

E. Cartan [2] proved that this space is a homogeneous Riemannian manifold $SO(3)/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ and its principal curvatures are equal to $\sqrt{3}$, 0, and $-\sqrt{3}$