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AN EXAMPLE OF RIEMANNIAN MANIFOLDS SATISFYING $R(X, Y)\cdot R = 0$ BUT NOT $\nabla R = 0$

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If a Riemannian manifold M is locally symmetric, then its curvature tensor R satisfies

$$(*) \qquad R(X, Y)\cdot R = 0 \text{ for all tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. Conversely, does this algebraic condition (*) on the curvature tensor field R imply that M is locally symmetric (i.e. $\nabla R = 0$)? For this problem, K. Nomizu conjectured that the answer is affirmative in the case where M is irreducible and complete and dim $M \geqq 3$.

In the present paper, we shall show that, in a 4-dimensional Euclidean space E^4, there exists an irreducible and complete hypersurface M which satisfies the condition (*) but is not locally symmetric.

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1. Reduction of condition (*). Let M be a 3-dimensional Riemannian manifold which is isometrically immersed in a Euclidean space E^4. Let U be a neighborhood of a point $p_0 \in M$ on which we can choose a unit vector field N normal to M. For any vector fields X and Y tangent to M, we have the formulas of Gauss and Weingarten:

$$(1.1) \qquad \begin{aligned} D_X Y &= \nabla_X Y + H(X, Y)N\,, \\ D_X N &= -AX\,, \end{aligned}$$

where D_X and ∇_X denote covariant differentiations for the Euclidean connection of E^4 and the Riemannian connection on M, respectively. A is a field of symmetric endomorphisms which corresponds to the second fundamental form H, that is, $H(X, Y) = g(AX, Y)$ for tangent vectors X and Y, g being the Riemannian metric induced from E^4. The equation of Gauss expresses the curvature tensor R of M by means of A:

$$R(X, Y)Z = g(Z, AY)AX - g(Z, AX)AY\,.$$

The type number $t(p)$ at $p \in M$ is, by definition, the rank of A at p. At a point $p \in M$, let $\{e_1, e_2, e_3\}$ be an orthonormal basis of the tangent