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MEAN VALUE THEOREMS FOR VECTOR-VALUED FUNCTIONS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. The object of this paper is to give a generalization to vector-valued functions of the Cauchy mean value theorem of the differential calculus, together with some related results. In the classical Cauchy mean value theorem we have

$$(\phi(b) - \phi(a))\psi'(\xi) = (\psi(b) - \psi(a))\phi'(\xi) \tag{1.1}$$

for some ξ in $]a, b[$, where $\phi, \psi\colon [a, b] \to \boldsymbol{R}$ are continuous functions possessing derivatives on $]a, b[$. The counterpart to (1.1) when ϕ is vector-valued is the mean value inequality in Theorem 1 below.

Throughout we suppose that our vector spaces are real. For any function ϕ from an interval $[a, b]$ into a topological vector space Y, we say that an element y of Y is a *right-hand derivative value* of ϕ at the point $t \in [a, b[$ if there exists a sequence (t_n) of points of $]t, b]$ decreasing to the limit t such that $(\phi(t_n) - \phi(t))/(t_n - t) \to y$ in Y as $n \to \infty$. (In particular, if $Y = \boldsymbol{R}$, a right-hand derivative value is finite.)

The use of right-hand derivative values* enables us to avoid the hypothesis that limits in Y are unique. However, if Y is a T_1-space (and therefore Hausdorff), we can define two-sided and one-sided derivatives in the usual way; for example, the right-hand derivative $\phi'_+(t)$ of ϕ at the point $t \in [a, b[$ is the limit

$$\lim_{h \to 0+} \frac{\phi(t + h) - \phi(t)}{h}$$

whenever this limit exists in Y (again it is finite if $Y = \boldsymbol{R}$). Obviously $\phi'_+(t)$ is a right-hand derivative value of ϕ at t.

A sublinear functional on a vector space Y is a function $p\colon Y \to \boldsymbol{R}$ such that for all $y, z \in Y$ and all $\lambda \geqslant 0$

$$p(y + z) \leqslant p(y) + p(z) \quad \text{and} \quad p(\lambda y) = \lambda p(y) . \tag{1.2}$$

We note in passing that the first of these relations implies that for all $y, z \in Y$

* The name is used by McLeod [12].