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FIXED POINT FREE INVOLUTIONS ON HOMOTOPY SPHERES

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1. Introduction. The original motivation for this work was the following idea. Suppose one could prove that if $T: S^n \to S^n$ is a PL involution without fixed points, then there exists a PL sphere $S^{n-1} \subset S^n$ such that $TS^{n-1} = S^{n-1}$. This would constitute the induction step in a proof that each fixed point free PL involution on S^n is conjugate, in the group of homeomorphisms of S^n, to the antipodal map, $\mathfrak{A}$. In the smooth case, one might try to argue in a similar way, but with due regard for the groups θ_n, [8].

This idea does not work. There are obstructions to finding a PL sphere S^{n-1} in S^n such that $TS^{n-1} = S^{n-1}$ when n is odd. When $n = 4k - 1$, there is a symmetric bilinear form whose index is determined by the pair (S^n, T) and which is the obstruction in this dimension. When $n = 4k + 1$, the bilinear form is skew-symmetric with an associated quadratic form ψ_0 (over Z_2). The obstruction in this case is the Arf invariant of ψ_0. Similar obstructions are encountered in trying to answer the following question: Suppose S_0^{n-1} and S_1^{n-1} are two spheres in (S^n, T) such that $TS_i^{n-1} = S_i^{n-1}$, $i = 0, 1$. Then are the involutions $(S_0^{n-1}, T|S_0^{n-1})$ and $(S_1^{n-1}, T|S_1^{n-1})$ equivalent? Here we call two involutions (S_1, T_1) and (S_2, T_2) PL or smoothly equivalent if there is an equivariant homeomorphism, PL or smooth, of (S_1, T_1) onto (S_2, T_2). This paper supplies the proofs of the theorems announced in [5]. Since the work described here was completed, (in 1965) much progress has been made in classifying fixed point free involutions, smooth or PL, on homotopy spheres. It does not seem appropriate to list here all of these works, especially since the thesis [9] of Santiago Lopez de Medrano contains a complete bibliography of this subject. Let it suffice to say that many of the obvious questions raised here have now been answered. See, in particular [9] and [21]. We shall work in the smooth category, but all of the results hold in the PL category.

2. Characteristic submanifolds. Let $T: W \to W$ be a smooth, fixed point free involution of the smooth manifold W. Denote the orbit space

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