Tôhoku Math. Journ.
25 (1973), 197-199.

ANALYTIC SUBGROUPS OF $GL(n, \boldsymbol{R})$

Dedicated to Professor Shigeo Sasaki on his 60th birthday

MORIKUNI GOTO

(Received July 1, 1972; Revised October 30, 1972)

1. Let L be a Lie group, let G be a connected Lie group, and let f be a continuous one-one homomorphism from G into L. Then the analytic structure of G is determined by the image $f(G)$. The set $f(G)$ with the Lie group structure of G is called an *analytic subgroup* of L. (See Chevalley [2].) $f(G)$ is not necessarily closed in L. Obviously, we can find continuous monomorphisms from $\boldsymbol{R}^m$ into a toral group of dimension greater than m. The purpose of this short note is to prove the following theorem, by which we can say, roughly speaking, that the above example exhausts all non-closed analytic subgroups in the case when L is the general linear group $GL(n, \boldsymbol{R})$.

For a subset M of $GL(n, \boldsymbol{R})$, $\overline{M}$ will denote the closure of M. The identity element of a group in question will always be denoted by e.

THEOREM. *Let G be a connected Lie group, and let f be a continuous one-one homomorphism from G into $GL(n, \boldsymbol{R})$. Then we can find a closed subgroup V, which is isomorphic with $\boldsymbol{R}^k$ for suitable $k = 0, 1, 2, \cdots$, and a connected closed normal subgroup N, such that G is a semi-direct product: $G = VN$, $V \cap N = e$. Here V and N may be selected so that $\overline{f(V)}$ is a toral group, $f(N)$ is closed, and $\overline{f(G)}$ is a local semi-direct product of $\overline{f(V)}$ and $f(N)$: $\overline{f(G)} = \overline{f(V)}f(N)$, and $\overline{f(V)} \cap f(N)$ is finite. Moreover, in this case $\overline{f(G)}$ is diffeomorphic with the direct product $\overline{f(V)} \times N$.*

2. Following a method in Borel [1], we shall first prove a lemma.

LEMMA. *Let L be a Lie group, and N a connected closed normal subgroup of L. Let T be a toral subgroup of L. If $L = TN$ and $T \cap N$ is finite, then the space of L is diffeomorphic with the product space $T \times N$.*

PROOF. First we note that it is enough to show the lemma for $\dim T = 1$. Then $L/N = T$ is a circle. On the other hand, a principal fibre bundle with a circle as its base space and with a connected Lie group as its fibre is always trivial. (A special case of Corollary 18.6 in Steenrod [5].) q.e.d.