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AN INEQUALITY BETWEEN SQUARE NORMS ON DUAL GROUPS

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Abstract. The Plancherel Theorem asserts the equality of the L^2-norms (with respect to Haar measure) of a function f on a locally compact abelian group G and of its Fourier transform $\hat{f}$. The Hausdorff-Young inequality gives conditions on p and q under which $||\hat{f}||_q \leqq ||f||_p$. We consider a different variant: we place a measure μ on $\hat{G}$, a measure w on G, and examine

$$\int_{\hat{G}} |\hat{f}|^2 d\mu \leqq \int_G |f|^2 dw .$$

Our main results show that it is enough to consider the case in which w is equivalent to Haar measure, and we give a condition on w which is necessary and sufficient for the inequality to hold for every $\mu \geqq 0$ with $||\mu|| \leqq 1$.

Let G be a locally compact abelian group and let $\hat{G}$ be its dual group. We denote the Fourier transform of a function f on G by $\hat{f}$. In this note we shall consider the inequality

$$\int_{\hat{G}} |\hat{f}|^2 d\mu \leqq \int_G |f|^2 dw \tag{1}$$

which we require to hold for all functions f in the space $\mathscr{K}(G)$ of continuous functions of compact support on G, for some positive measures μ on $\hat{G}$ and w on G.

Inequalities of this kind have a long history (see, for example [2]). They have appeared more recently because of their importance in the solution of multiplier problems for weighted L^p-spaces ([5], in particular a remark on page 50, and [6], especially Lemmas 2.1 and 2.2). These authors usually consider cases in which one of the groups G and $\hat{G}$ is the circle group and the other the integers, though in his Theorem 3b in [5], Hirschman quotes a result for R^n. Work on general groups has usually yielded only abstract characterizations of multipliers [1], and we hope that a study of the inequality (1) might be a first step to some more concrete representations.

Our principal results are as follows. First, if the inequality (1) holds for some non-zero measure μ, then the Haar measure m of G must be absolutely continuous with respect to w. Moreover, the inequality remains