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HARMONIC AND QUASIHARMONIC DEGENERACY OF RIEMANNIAN MANIFOLDS

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The harmonic and quasiharmonic classifications of Riemannian manifolds have been largely brought to completion (see Bibliography). In the present paper we shall discuss interrelations between harmonic null classes and quasiharmonic null classes.

Let H and Q be the classes of harmonic and quasiharmonic functions h and q, defined by $\Delta h = 0$ and $\Delta q = 1$, where $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator. Denote by P, B, D, C, L^p the classes of functions which are positive, bounded, Dirichlet finite, bounded Dirichlet finite, or of finite L^p norm, respectively. Here $1 \leqq p < \infty$, the value $p = \infty$ being excluded since for both harmonic and quasiharmonic functions, $L^\infty = B$. For $X = P, B, D, C, L^p$, set $HX = H \cap X$, $QX = Q \cap X$, and let O_F^N stand for the class of Riemannian N-manifolds, $N \geqq 2$, which do not carry nonconstant functions in a given class F. The complement of O_F^N with respect to the totality of Riemannian N-manifolds is designated by $\tilde{O}_F^N$.

We shall first show that $O_{HX}^N \cap O_{QY}^N \neq \emptyset$ for $X, Y = P, B, D, C, L^p$, $1 \leqq p < \infty$, $N \geqq 2$. In view of the Euclidean ball it is trivial that $\tilde{O}_{HX}^N \cap \tilde{O}_{QY}^N \neq \emptyset$, and we shall prove that $\tilde{O}_{HX}^N \cap O_{QY}^N \neq \emptyset$ for all X, Y, p, N.

The classes $O_{HX}^N \cap \tilde{O}_{QY}^N$ are intriguing. From the harmonic classification theory it is known that the class O_G^N of parabolic N-manifolds, characterized by the nonexistence of Green's functions, is related to other harmonic null classes by the strict inclusion relations $O_G^N < O_{HP}^N < O_{HB}^N < O_{HD}^N = O_{HC}^N$. The proof of the strictness, due mainly to Ahlfors, Royden, and Tôki, was one of the most challenging problems in the theory of harmonic functions. On the other hand, O_G^N is strictly contained also in all O_{QX}^N, $X = P, B, D, C$ [12, 20]. The problem of proving the nonemptiness of the classes $O_{HX}^N \cap \tilde{O}_{QY}^N$ thus amounts to finding manifolds which belong to the "narrow" spaces $\tilde{O}_G^N \cap O_{HX}^N$, yet carry QY-functions. For X, Y other than L^p we only have fragmentary results on this problem (see No. 11). On the other hand, the classes $O_{HX}^N \cap \tilde{O}_{QY}^N$ turn out to be

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