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A NOTE ON THE VON NEUMANN ALGEBRA WITH A CYCLIC AND SEPARATING VECTOR

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Let M be a von Neumann algebra on a Hilbert space H with a cyclic and separating vector. If, for some cyclic and separating vector ξ_0 in H for M,

$$M\xi_0 = M'\xi_0 , \tag{$*$}$$

then we shall call that M has the property (J).

Note that M satisfying the equality $(*)$ for ξ_0 does not always imply for the other cyclic and separating vector.

In [Theorem 1] we show that M has the property (J) if and only if M is a finite von Neumann algebra. In terms of the Hilbert algebra, we can consider $M\xi_0$ as an achieved left Hilbert algebra $\mathfrak{A}$ with the product: $(x\xi_0)(y\xi_0) = xy\xi_0$, and the involution: $S(x\xi_0) = x^*\xi_0$, $x, y \in M$, and $M'\xi_0$ as the right Hilbert algebra $\mathfrak{A}'$ of $\mathfrak{A}$. (see [2], [3]) The analysis in this paper may be the special case of the characterization of the type of the left von Neumann algebra $\mathfrak{L}(\mathfrak{A})$ associated to the achieved left Hilbert algebra $\mathfrak{A}$ under the condition $\mathfrak{A} = \mathfrak{A}'$ as a set. Without difficulty, we can prove that an achieved left Hilbert algebra $\mathfrak{A}$ is equal to $\mathfrak{A}'$ as a set if and only if $\mathfrak{A}$ is a Tomita algebra.

In [Theorem 2] we shall give a characterization of a finite von Neumann algebra via the Radon-Nikodym theorem for the state. We mainly refer [1] and [2].

Now, we state here that if M is finite, then M has the property (J).

In fact, let ξ_0 be a cyclic and separating trace vector in H for M. Then we have

$$||S(x\xi_0)||^2 = ||x^*\xi_0||^2 = (xx^*\xi_0|\xi_0) = (x^*x\xi_0|\xi_0) = ||x\xi_0||^2 ,$$

for all x in M. Therefore $M\xi_0$ is a uni-modular Hilbert algebra. From [2] Cor. 10.1, we have $M\xi_0 = M'\xi_0$.

Now we need the following lemma to prove [Theorem 1].

LEMMA (cf. [1] Chap. I §1 ex. 5). *Suppose that M is a von Neumann algebra on a Hilbert space H such that $M\xi_0 = M'\xi_0$ for a cyclic and separating vector ξ_0 in H, that is, for any element x in M, there exists*