Tôhoku Math. Journ.
31 (1979), 179–185.

THE FIRST EIGENVALUE OF THE LAPLACIAN ON SPHERES

SHÛKICHI TANNO

(Received April 25, 1978, revised May 19, 1978)

1. Introduction. Let (S^2, h) be a 2-dimensional sphere with metric h, and let $\lambda_0 = 0 < \lambda_1 = \lambda_1(h) \leqq \lambda_2 \leqq \cdots$ be eigenvalues of the Laplacian Δ on (S^2, h) acting on smooth functions. J. Hersch [3] showed that

$$(*) \qquad 1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 \geqq (3/8\pi)\mathrm{Vol}(S^2, h)$$

holds, and in particular

$$(**) \qquad \lambda_1(h)\mathrm{Vol}(S^2, h) \leqq 8\pi ,$$

where $\mathrm{Vol}(S^2, h)$ denotes the volume of S^2 with respect to h. Equality in $(*)$ or $(**)$ holds if and only if h is a constant curvature metric.

M. Berger [1] showed that $(*)$ cannot be generalized for (S^m, h), $m \geqq 3$. With respect to $(**)$, M. Berger [1] posed a problem: Let M be a compact smooth manifold; then does there exist a constant $k(M)$ depending only on M such that the first eigenvalue $\lambda_1(h)$ of the Laplacian satisfies

$$(***) \qquad \lambda_1(h)\mathrm{Vol}(M, h)^{2/m} \leqq k(M)$$

for any Riemannian metric h?

H. Urakawa [5] showed the following: Let G be a compact connected Lie group with a non-trivial commutator subgroup; then there exists a family of left invariant Riemannian metrics $g(t)$ $(0 < t < \infty)$ on G such that

$$(****) \qquad \begin{cases} \lambda_1(g(t)) \to \infty & \text{as} \quad t \to \infty , \\ \lambda_1(g(t)) \to 0 & \text{as} \quad t \to 0 \end{cases}$$

and $\mathrm{Vol}(G, g(t)) = \text{constant}$. In particular, since $SU(2)$ is diffeomorphic to S^3, there exists no constant $k(S^3)$ for a 3-dimensional sphere S^3 such that $(***)$ holds.

The purpose of this paper is to prove that for any odd dimensional sphere S^{2n+1} there exists no constant $k(S^{2n+1})$ such that $(***)$ holds. Namely we show the following.

THEOREM. *Any odd dimensional sphere* S^{2n+1}, $n \geqq 1$, *admits a family of Riemannian metrics* $g(t)$ $(0 < t < \infty)$ *such that the first*