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ON TRANSFORMING THE CLASS OF BMO-MARTINGALES BY A CHANGE OF LAW

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1. **Introduction.** If Z is a positive uniformly integrable martingale such that $Z_0 = 1$, then we can define a change of the underlying probability measure dP by the formula $d\hat{P} = Z_\infty dP$. Our interest in this paper lies in investigating the transformation of BMO-martingales by this change of law. Let us denote by $B(P)$ (resp. $B(\hat{P})$) the space of BMO-martingales with respect to dP (resp. $d\hat{P}$). In the next section we shall deal only with discrete time martingales, and prove that $B(\hat{P})$ is isomorphic to $B(P)$ under a certain assumption. This equivalence corresponding to the continuous time case will be established in Section 4. Furthermore, in Section 3, we shall give a characterization of BMO-martingales.

2. **The equivalence of $B(P)$ and $B(\hat{P})$; the discrete time case.** Let (Ω, F, P) be a probability space, given a non-decreasing sequence (F_n) of sub σ-fields of F such that $\bigvee_{n=1}^{\infty} F_n = F$. We shall assume that F_0 contains all null sets. If $X = (X_n, F_n)$ is a martingale with difference sequence $x = (x_n)_{n\geqq 1}$, then the square function of X is $S(X) = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$. Let $S_n(X) = (\sum_{k=1}^{n} x_k^2)^{1/2}$, $S_0(X) = X_0 = 0$ and if X converge a.s., let X_∞ denote its limit. The reader is assumed to be familiar with the martingale theory as is given in [2] and [3]. Throughout the paper, let us denote by C a positive constant and by C_p a positive constant depending only on the indexed parameter p, both letters are not necessarily the same in each occurrence. X is a BMO-martingale if

$$||X||_{B(P)} = \sup_n || E[S(X)^2 - S_{n-1}(X)^2 | F_n]^{1/2} ||_\infty < \infty .$$

The class of BMO-martingales depends on the underlying probability measure and so we shall denote it by $B(P)$. It is a real Banach space with norm $||\cdot||_{B(P)}$. The next lemma is fundamental in our investigation.

LEMMA 1. *The inequality*

$$E[\exp\{S(X)^2 - S_{n-1}(X)^2\} | F_n] \leqq (1 - ||X||_{B(P)}^2)^{-1} \tag{1}$$

is valid for every martingale X such that $||X||_{B(P)} < 1$.