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CHANGES OF LAW, MARTINGALES AND THE CONDITIONED SQUARE FUNCTION

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Let (Ω, F, P) be a complete probability space, given an increasing sequence (F_n) of sub σ-fields of F such that $F = \bigvee_{n\geqq 0} F_n$. If $f = (f_n, F_n)$ is a martingale with difference sequence $d = (d_n)_{n\geqq 1}$, we shall set $f^* = \sup_{n\geqq 0}|f_n|$, $S(f) = (\sum_{n=1}^{\infty} d_n^2)^{1/2}$ and $s(f) = (\sum_{n=1}^{\infty} E[d_n^2|F_{n-1}])^{1/2}$. Let us assume that $f_0 = 0$. The operator $s(f)$, which is not of matrix type, is called the conditioned square function. It was studied by Burkholder and Gundy [3]. Let $s_n(f) = (\sum_{k=1}^{n} E[d_k^2|F_{k-1}])^{1/2}$. Clearly, $s_n(f)$ is F_{n-1}-measurable. Throughout the paper, we fix a BMO-martingale $M_n = \sum_{k=1}^{n} m_k$, $M_0 = 0$ such that $-1 + \delta < m_k$, $(k \geqq 1)$ for some constant δ with $0 < \delta \leqq 1$, and consider the process Z given by the formula $Z_n = \prod_{k=1}^{n}(1 + m_k)$, $Z_0 = 1$. Z is a positive uniformly integrable martingale which satisfies the condition

$$(A_p) \qquad Z_n E[Z_\infty^{-1/(p-1)}|F_n]^{p-1} \leqq C_p\,, \quad n \geqq 0$$

for some $p > 1$; see [6]. As $Z_\infty > 0$ a.s., the weighted probability measure $d\hat{P} = Z_\infty dP$ is equivalent to dP. Note that for every $\hat{P}$-integrable random variable Y

$$\hat{E}[Y|F_n] = E[Z_\infty Y|F_n]/Z_n \qquad \text{a.s., under } dP \text{ and } d\hat{P}\,,$$

where $\hat{E}$ denotes the expectation over Ω with respect to $d\hat{P}$.

Our aim is to prove the following:

THEOREM. *Let $0 < p \leqq 2$. Then the inequality*

$$(1) \qquad \hat{E}[(f^*)^p] \leqq c_p \hat{E}[s(f)^p]$$

is valid for all martingales $f = (f_n)$.

Furthermore, if $2 \leqq p < \infty$ and Z satisfies the (A_p) condition, then we have

$$(2) \qquad \hat{E}[s(f)^p] \leqq C_p \sup_{n\geqq 0} \hat{E}[|f_n|^p]\,.$$

Here, the choice of c_p and C_p depends only on p.

This result is well-known for the case where $Z \equiv 1$; see Theorem 5.3 of [3]. To prove the theorem, we need several lemmas, which will