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ALMOST SURE INVARIANCE PRINCIPLES FOR LACUNARY TRIGONOMETRIC SERIES

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1. Introduction. In this note let $\{n_m\}$ be a sequence of positive integers satisfying the gap condition

$$(1.1) \qquad n_{m+1}/n_m > 1 + cm^{-\alpha} \qquad (c > 0 \text{ and } 0 \leqq \alpha \leqq 1/2)\,,$$

and $\{a_m\}$ be a sequence of positive numbers such that

$$(1.2) \qquad \begin{cases} A_k = \left(2^{-1}\sum_{m=1}^{k} a_m^2\right)^{1/2} \to +\infty\,, \\ a_k = O(A_k k^{-\alpha}(\log A_k)^{-\beta})\,, \qquad \beta > 1/2\,, \qquad \text{as} \quad k \to +\infty\,. \end{cases}$$

Further, we put

$$(1.3) \qquad \xi_m(\omega) = a_m \cos 2\pi(n_m\omega + \alpha_m) \quad \text{and} \quad T_k = \sum_{m=1}^{k} \xi_m\,,$$

where $\{\alpha_m\}$ is a sequence of arbitrary real numbers, and consider ξ_m's as random variables on a probability space $([0, 1), \mathscr{F}, P)$ where $\mathscr{F}$ is the σ-field of all Borel sets on $[0, 1)$ and P is the Lebesgue measure on $\mathscr{F}$. Then we write, for $\omega \in [0, 1)$ and $t \geqq 0$,

$$(1.4) \qquad S(t) = S(t, \omega) = T_k(\omega)\,, \qquad \text{if} \quad A_k^2 \leqq t < A_{k+1}^2\,,$$

for $k \geqq 0$, where we put $A_0 = 0$ and $T_0 = 0$.

The purpose of the present paper is to prove the following.

THEOREM. *Without changing the distribution of $\{S(t), t \geqq 0\}$ we can redefine the process $\{S(t), t \geqq 0\}$ on a richer probability space together with standard Brownian motion $\{X(t), t \geqq 0\}$ such that*

$$S(t) = X(t) + o(t^{1/2}) \quad a.s. \qquad as \ t \to +\infty\,.$$

Using the almost sure limiting behavior of $\{X(t), t \geqq 0\}$ and the above theorem we can deduce the corresponding limiting properties of $\{S(t), t \geqq 0\}$ or $\{T_k(\omega)\}$. For example we can obtain the following

COROLLARY (cf. [3]). *Under the conditions* (1.1) *and* (1.2) *we have, for a.e. ω,*