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POSITIVELY INVARIANT SETS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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1. Introduction. For ordinary differential equations, many authors have discussed necessary and sufficient conditions for a closed set in the n-dimensional Euclidean space R^n to be positively invariant. Yorke [11] has discussed this problem by using a non-Lipschitzian Liapunov function which is lower-semicontinuous. For an autonomous system, Brezis [1] obtained a result under the assumption that the right hand side of the system is locally Lipschitzian, and his proof depends essentially on this assumption. Crandall [2] obtained a similar result by applying the method of polygonal approximations. For a nonautonomous system, Hartman [5] also considered an approximation which is different from the one considered in [2].

The purpose of this article is to discuss the same question for functional differential equations with infinite delay. Seifert [10] also discussed this question under the assumption that a closed set is convex. In Section 2, we introduce an abstract phase space B which satisfies some general hypotheses slightly different from those considered in [4]. We consider a subset Ω in $R \times R^n$ such that the cross section $\Omega_t = \{y \in R^n; (t, y) \in \Omega\}$ is convex for all $t \in R$ and that the cross section Ω_t satisfies a continuity condition in the sense of Hausdorff metric. We discuss the properties of Ω which play an important role in Section 3. In Section 3, we state the main theorem. We give the necessary and sufficient condition that, for any initial value (σ, ϕ) in $R \times B$ such that $\phi(t - \sigma) \in \Omega_t$ for all $t \leqq \sigma$, there exists at least one solution $x(t)$ through (σ, ϕ) which is defined on its right maximal interval of existence and satisfies $(t, x(t)) \in \Omega$ there. Special approximate solutions are needed to prove the theorem. The construction of the solutions, although analogous to the one in [5], is much more complicated for functional differential equations. The proof of the theorem is given in Section 4. The case where the delay is finite has been considered in [7] and [8] by a different approach.