Tôhoku Math. Journ.
34 (1982), 499–507.

ON YAMABE'S PROBLEM —BY A MODIFIED DIRECT METHOD—

ATSUSHI INOUE

(Received December 7, 1981)

1. Introduction. In his paper [12], Yamabe asked whether any compact C^∞-Riemannian manifold of dimension $\geqq 3$ can be deformed conformally to a C^∞-Riemannian manifold of constant scalar curvature, and proposed to solve the question by reducing it to a non-linear eigenvalue problem which consists of finding a strict positive C^∞ function u on a compact Riemannian manifold (M, g) together with a constant λ so that one has

$$-(4(n-1)/(n-2))\Delta u + Ru = \lambda u^{(n+2)/(n-2)} \tag{1.1}$$

where Δ is the usual Laplace-Beltrami operator and $R = R(x)$ denotes the scalar curvature defined by the metric g.

As pointed out by Trudinger [11] there was a gap in Yamabe's proof, and the problem is still unsolved as it stands. An almost complete solution, however, has recently been achieved by Aubin [1], [2]. In fact, introducing the functional

$$J(u) = \int_M (\kappa|\nabla u|^2 + Ru^2)d\omega\Big/\Big(\int_M |u|^N d\omega\Big)^{2/N}\,, \qquad \kappa = 4(n-1)/(n-2)\,, \quad N = 2n/(n-2) \tag{1.2}$$

where ∇ denotes the covariant derivation, $d\omega$ the volume element relative to the metric g, and the number

$$\mu = \inf\{J(u);\, u \in H^1(M),\, u \neq 0\} \tag{1.3}$$

here $H^1(M)$ implying as usual the Sobolev space of degree one, he proved that for any compact Riemannian manifold of dimension $\geqq 3$ there holds an inequality $\mu \leqq n(n-1)\omega_n^{2/n}$ with ω_n denoting the surface area of n-sphere S^n, and, among others, the following:

THEOREM. *If $\mu < n(n-1)\omega_n^{2/n}$, there exists a strictly positive C^∞ function on (M, g) satisfying* (1.1) *with $\lambda = \mu$. (This gives a partial answer to Yamabe's problem.)*

The purpose of this paper is to give a simplified proof of the above result, using a different approach from that of Aubin, but similar in