Tôhoku Math. Journ.
35 (1983), 187-224.

AN ALGEBRAIC APPROACH TO ISOPARAMETRIC HYPERSURFACES IN SPHERES I

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(Received November 26, 1981)

Introduction. An isoparametric hypersurface in a sphere is an orientable submanifold of the sphere which has codimension 1 and constant principal curvatures. Cartan was the first to study such hypersurfaces e.g. [1]. The subject seems to have been forgotten till it was revived by Nomizu, who published a survey on E. Cartan's theory of isoparametric hypersurfaces [9]. Takagi and Takahashi applied results of Hsiang and Lawson on orbits of codimension 1 to classify all homogeneous hypersurfaces in spheres [12]. This classification includes the description of all hypersurfaces with at most 3 distinct principal curvatures since Cartan had shown that all such hypersurfaces are homogeneous. In [7] and [8], Münzner proved that the number g of distinct principal curvatures of an isoparametric hypersurface in a sphere is 1, 2, 3, 4 or 6. Moreover, refining ideas of Cartan, he showed that each such hypersurface is an open submanifold of a level surface of a homogeneous polynomial of degree g and characterized these polynomials by two differential equations. Obviously, it remains to consider the cases $g = 4$ and $g = 6$ and to classify the corresponding polynomials. Of course, this would be superfluous if all isoparametric hypersurfaces in a sphere were homogeneous. As mentioned above, for $g = 1, 2, 3$ all hypersurfaces are homogeneous. However, there exist non-homogeneous examples. The first non-homogeneous examples were found by Ozeki and Takeuchi [10], [11]. They constructed two infinite series of non-homogeneous isoparametric hypersurfaces. Recently, Ferus, Karcher, and Münzner found— for $g = 4$—a new type of examples (constructed from representations of a Clifford algebra) which includes all known non-homogeneous examples and—with the exception of two manifolds—all homogeneous examples [6]. They even constructed infinitely many infinite series of non-homogeneous hypersurfaces.

In this paper we develop a new algebraic approach to isoparametric hypersurfaces in spheres. We concentrate on the case $g = 4$, but the

* Partially supported by NSF Grant MCS-8104848.
** Supported by Deustsche Forschungsgemeinschaft.