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GENERALIZED INVERSE METHOD FOR SUBSPACE MAPS

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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1. Introduction. Let H be a Hilbert space and let $C(H)$ be the set of all closed linear subspaces in H. For a bounded linear operator A on H, define a map ϕ_A on $C(H)$, called the subspace map of A, by

$$\phi_A(M) = (AM)^- \qquad (M \in C(H)) ,$$

where "$-$" denotes the uniform closure. Identifying every closed subspace M with the corresponding (orthogonal) projection P_M or $\operatorname{proj} M$, we see that $C(H)$ is a subset of $B(H)$, the Banach space of all bounded linear operators on H and hence has the uniform, strong and weak (operator) topologies. It was shown in [8] (cf. [2]) that the subspace map ϕ_A is uniformly (and strongly) continuous on $C(H)$ if and only if the operator A is left-invertible, and moreover, in this case ϕ_A behaves well. For instance, $\phi_A(\mathscr{F})$ is uniformly (resp. strongly, weakly) closed if $\mathscr{F}$ is a uniformly (resp. strongly, weakly) closed subset of $C(H)$.

In this paper we shall show similar results on the subspace map ϕ_A under the weaker condition that the operator A has closed range, or equivalently, has the (Moore-Penrose) generalized inverse [1] [9]; using operator theory of generalized inverses, we shall discuss the local continuity and some other topological properties of ϕ_A of A with closed range, which will extend some results in [2] and [8].

Throughout this note we shall write $A \in (\mathrm{CR})$ when the operator A has closed range. The generalized inverse $A^\dagger$ of $A \in (\mathrm{CR})$ satisfies (and is determined by) the following four Penrose identities [1]

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A .$$

If we denote by AH and $\ker A$ the range and the kernel of $A(\in(\mathrm{CR}))$ respectively, then the products $AA^\dagger$ and $A^\dagger A$ represent the projections onto AH and the orthogonal complement $(\ker A)^\perp$ of $\ker A$ respectively [1]. For two projections P and Q, write $P^\perp$ and $P \vee Q$ for the projection onto $(PH)^\perp$ and for that onto the closed linear span of PH and QH, respectively. Now, for our later discussion we state three lemmas on