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ON THE FUNCTIONS OF LITTLEWOOD-PALEY AND MARCINKIEWICZ

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1. Introduction. Let $f(x)$ be a locally integrable function on the real line $\boldsymbol{R}$. The Fourier integral analogue of Marcinkiewicz function [7] is

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 t^{-3} dt\right)^{1/2}$$

where

$$F(x) = \int_0^x f(u)du .$$

We generalize this as follows: for $\alpha > 0$

$$\mu_\alpha(f)(x) = \left\{\int_0^\infty \left|\frac{\alpha}{t}\int_0^\infty \left(1 - \frac{u}{t}\right)^{\alpha-1} (f(x-u) - f(x+u))du\right|^2 \frac{dt}{t}\right\}^{1/2}, \tag{1.1}$$

$\mu_1(f)(x)$ coincides with $\mu(f)(x)$. (1.1) is the one dimensional form of the more general Marcinkiewicz function

$$\mu(f)(x) = \left\{c\int_0^\infty \left|\frac{1}{t}\int_{|u|\leq t} f(x-u)\frac{\Omega(u')}{|u|^{k-1}}du\right|^2 \frac{dt}{t}\right\}^{1/2}$$

where $\Omega(u')/|u|^k$ is the Calderón-Zygmund kernel on k-dimensional space and c is a constant depending on k only, see Stein [8].

On the other hand we have generalized the Littlewood-Paley function as follows

$$g_\beta^*(\phi)(x) = \left\{\frac{1}{\pi}\int_0^\infty y^{2\beta}dy \int_{-\infty}^\infty \frac{|\phi'(t+iy)|^2}{|t-x-iy|^{2\beta}}dt\right\}^{1/2}, \tag{1.2}$$

where $\phi(z) = \phi(x+iy)$ is analytic in the upper half-plane and has boundary value $\phi(x) = \lim_{y\to 0}\phi(x+iy)$. The original Littlewood-Paley function $g^*(\phi)(x)$ in Fourier integral form corresponds to the case $\beta = 1$ in (1.2).

Let $\sigma_\beta(R; x, \beta)$ the R-th (C, β)-mean of Fourier integral of complex valued function $\phi(x)$ and set

$$\tau_\beta(R; x, \phi) = R\frac{d}{dR}\sigma_\beta(R; x, \phi) = \beta\{\sigma_{\beta-1}(R; x, \phi) - \sigma_\beta(R; x, \phi)\} \tag{1.3}$$

and set