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INFINITESIMAL ISOMETRIES OF FRAME BUNDLES WITH NATURAL RIEMANNIAN METRIC

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1. Introduction. Let $(M, \langle\ ,\ \rangle)$ be a connected orientable Riemannian manifold of dimension $n \geqq 3$ and $SO(M)$ be the bundle of all oriented orthonormal frames over M. $SO(M)$ has a Riemannian metric, also denoted by $\langle\ ,\ \rangle$, defined naturally as follows: At each point u of $SO(M)$, the tangent space $SO(M)_u$ is a direct sum $Q_u + V_u$, where Q_u is the horizontal space defined by the Riemannian connection and V_u is the space of vectors tangent to the fibre through u. The right action of the special orthogonal group $SO(n)$ on the bundle $SO(M)$ gives an isomorphism f_u of the Lie algebra $\mathfrak{o}(n)$ onto V_u for each $u \in SO(M)$. We denote by A_u the image of $A \in \mathfrak{o}(n)$. On the other hand, $SO(n)$ has a bi-invariant metric denoted also by $\langle\ ,\ \rangle$, which is defined by

$$\langle A, C\rangle = \sum_{i,j} A_{ij} C_{ij}\,, \qquad A, C \in \mathfrak{o}(n)\,.$$

Then, the Riemannian metric $\langle\ ,\ \rangle$ of $SO(M)$ is defined by

$$\begin{aligned}
&\langle A_u, C_u\rangle = \langle A, C\rangle \\
&\langle A_u, X_u\rangle = 0 \\
&\langle X_u, Y_u\rangle = \langle pX_u, pY_u\rangle
\end{aligned}$$

for $X_u, Y_u \in Q_u$ and $A, C \in \mathfrak{o}(n)$, where p is the projection $SO(M) \to M$.

O'Neill [4] studied the curvature of $(SO(M), \langle\ ,\ \rangle)$. In the present paper, we shall study Killing vector fields on $(SO(M), \langle\ ,\ \rangle)$ and prove the following Theorems A and B. Let X be a vector field on $SO(M)$. X is said to be vertical (resp. horizontal) if $X_u \in V_u$ (resp. if $X_u \in Q_u$) for all $u \in SO(M)$. X is said to be fibre preserving if $[X, X']$ is vertical for any vertical vector field X'. Let A^* be the vertical vector field defined by $(A^*)_u = A_u = f_u(A)$. A^* is called the fundamental vector field corresponding to $A \in \mathfrak{o}(n)$. X is decomposed uniquely as $X = X^H + X^V$, with X^H horizontal and X^V vertical. X^H and X^V are called th horizontal part and the vertical part of X, respectively. Let ϕ be a 2-form on M. Then the tensor field F of type (1,1) is defined by $\langle FY, Z\rangle = \phi(Y, Z)$. Then, for each $u \in SO(M)$, $F^\sharp(\tilde{u}) \in \mathfrak{o}(n)$ is defined by

$$F^\sharp(u) = u^{-1} \circ F_{p(u)} \circ u\,,$$

where u is regarded as a linear isometry of $(\boldsymbol{R}^n, \langle\ ,\ \rangle)$ onto the tangent space $M_{p(u)}$ at $p(u)$. Here $\langle\ ,\ \rangle$ also denotes the standard metric of $\boldsymbol{R}^n$. Then, the vertical vector field