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GRÖBNER BASES OF TORIC VARIETIES

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Abstract. In this article projective toric varieties are studied from the viewpoint of Gröbner basis theory and combinatorics. We characterize the radicals of all initial ideals of a toric variety $X_{\mathscr{A}}$ as the Stanley-Reisner ideals of regular triangulations of its set of weights $\mathscr{A}$. This implies that the secondary polytope $\Sigma(\mathscr{A})$ is a Minkowski summand of the state polytope of $X_{\mathscr{A}}$. Here the lexicographic (resp. reverse lexicographic) initial ideals of $X_{\mathscr{A}}$ arise from triangulations by placing (resp. pulling) vertices. We also prove that the state polytope of the Segre embedding of $\boldsymbol{P}^{r-1}\times\boldsymbol{P}^{s-1}$ equals the secondary polytope $\Sigma(\Delta_{r-1}\times\Delta_{s-1})$ of a product of simplices.

1. Introduction. A cornerstone for the interaction between combinatorics and algebraic geometry is the theory of toric varieties [8], [16] which relates algebraic torus actions to the combinatorial study of convex polytopes. In the present paper we investigate the class of projective toric varieties from the point of view of Gröbner basis theory [1], [7], [11], [15], [20], [21]. The methods used to study Gröbner bases here are combinatorial rather than algebraic. Recent results on regular triangulations and secondary polytopes [5], [10], [13] will be applied to describe, as explicitly as possible, the initial ideals with respect to all term orders of a given projective toric variety.

Let $\mathscr{A}=\{a_1, a_2, \cdots, a_n\}$ be a fixed subset of the lattice $\boldsymbol{Z}^{d-1}\times\{1\}$ with the property that $\mathscr{A}$ linearly spans $\boldsymbol{R}^d$. A diagonal action of the d-dimensional torus $(\boldsymbol{C}^*)^d$ on $\boldsymbol{C}^n$ is obtained by interpreting $\mathscr{A}$ as the set of weights. Since the a_i all lie in an affine hyperplane, we get an induced $(\boldsymbol{C}^*)^d$-action on projective space $\boldsymbol{P}^{n-1}$. We define the *projective toric variety* $X_{\mathscr{A}}$ to be the closure of the orbit $(\boldsymbol{C}^*)^d\cdot(1, 1, \cdots, 1)$ in $\boldsymbol{P}^{n-1}$. The vanishing ideal of $X_{\mathscr{A}}$ is the homogeneous prime ideal

$$\mathscr{I}_{\mathscr{A}}:=\mathrm{Kernel}(\boldsymbol{C}[y_1,y_2,\cdots,y_n]\to\boldsymbol{C}[x_1,\cdots,x_d,x_1^{-1},\cdots,x_d^{-1}],\quad y_i\mapsto x^{a_i}=x_1^{a_{i1}}x_2^{a_{i2}}\cdots x_d^{a_{id}}).$$

Note that here we allow $\mathscr{A}$ to be any set of lattice points, which means that the embedded toric variety $X_{\mathscr{A}}$ need not be projectively normal.

We shall be interested in Gröbner bases of the *toric ideal* $\mathscr{I}_{\mathscr{A}}$ with respect to an arbitrary term order on $\boldsymbol{C}[y]:=\boldsymbol{C}[y_1,y_2,\cdots,y_n]$. In Section 2 we prove a singly-exponential (in d) degree upper bound for these Gröbner bases and thus for the Castelnuovo regularity of $\mathscr{I}_{\mathscr{A}}$. In addition, we construct an explicit universal Gröbner basis $\mathscr{U}_{\mathscr{A}}$ which satisfies this bound.

Our main result is a natural correspondence, to be established in Section 3, between the distinct Gröbner bases of $\mathscr{I}_{\mathscr{A}}$ and the regular triangulations of $\mathscr{A}$. As a corollary we find that the secondary polytope of $\mathscr{A}$ is a Minkowski summand of the state polytope