Tôhoku Math. J.
43 (1991), 213–234

A LOCALIZATION THEOREM FOR $\mathscr{D}$-MODULES

MUTSUMI SAITO

(Received February 26, 1990, revised May 31, 1990)

Introduction. Atiyah [1] invented equivariant K-theory, and he proved the so-called *Atiyah-Bott character formula* in collaboration with Bott [2]. Here we try to generalize the Atiyah-Bott character formula to the $\mathscr{D}$-module case, when an algebraic torus T acts on an algebraic variety X.

There are two finiteness conditions for $\mathscr{D}$-modules, namely, coherency and holonomicity. We adopt an approach à la Grothendieck. Therefore we choose holonomicity as a finiteness condition, since holonomicity is preserved under pull-backs and push-forwards.

We should notice that the concept of weakly equivariant $\mathscr{D}$-modules in [5] is better suited for our purposes rather than that of (strongly) equivariant $\mathscr{D}$-modules, because the Grothendieck group of weakly T-equivariant holonomic $\mathscr{D}$-modules on a point is the representation ring $R(T)$ of T, whereas that of (strongly) T-equivariant holonomic $\mathscr{D}$-modules on a point is just $\boldsymbol{Z}$. Therefore we consider the Grothendieck group $K_{T,h}(X)$ of weakly T-equivariant holonomic $\mathscr{D}_X$-modules. However, a difficulty lies in the fact that the nontrivial $R(T)$-module $K_{T,h}(T)$ is free. In order to avoid this difficulty, we put more relations into the Grothendieck group $K_{T,h}(X)$. Namely, we suppose that we have a relation when there exists an exact sequence as T-equivariant $\mathscr{O}_X$-modules. This quotient group of $K_{T,h}(X)$ is denoted by $\tilde{K}_T(X)$.

In Section 1, we develop generalities on weakly equivariant $\mathscr{D}$-modules which are discussed in [10]. In Section 2, we prove the existence of good stratifications, which are appropriate for our purposes. In Section 3, we prove a localization theorem for $\tilde{K}_T(X)$:

THEOREM. *The homomorphism of $R(T)_\Lambda$-modules*

$$(\boldsymbol{R}\Gamma_{X^T})_\Lambda : \tilde{K}_T(X)_\Lambda \longrightarrow \tilde{K}_T(X)_\Lambda$$

is the identity, where Λ is the multiplicatively closed subset of $R(T)$ generated by $1-\chi$ for nontrivial characters χ of T.

In Section 4, we define a formal character morphism

$$\mathrm{ch} : \tilde{K}_T(X) \longrightarrow R(T)_\Lambda ,$$

when a smooth T-variety X satisfies a *positivity condition*. Then we obtain an Atiyah-Bott type character formula for holonomic $\mathscr{D}$-modules as a corollary of the localization