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THE CLOSURES OF NILPOTENT ORBITS IN THE CLASSICAL SYMMETRIC PAIRS AND THEIR SINGULARITIES

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0. Introduction. Let G be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and θ an involution of G as an algebraic group. We also denote by θ the induced involution of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to θ, K_θ the subgroup of G consisting of elements $g\in G$ such that $\theta(g)=g$ and $N(\mathfrak{p})$ the nilpotent subvariety of $\mathfrak{p}$. We call the pair $(\mathfrak{g}, \mathfrak{k})$ the symmetric pair defined by (G, θ).

For the symmetric pairs, Sekiguchi [Se1] tried to construct an analogue of the Brieskorn-Slodowy theory ([B], [Sl]) which gives a correspondence between the simple Lie algebras and the rational double points. In [Se1], he introduced the problem to determine the generic singularities in $N(\mathfrak{p})$. To determine the generic singularities, we need the classification of K_θ-orbits in $N(\mathfrak{p})$ and their closure relation. In the classical cases, the classification of nilpotent orbits is given by means of *ab*-diagrams in [O2]. The first purpose of this paper is to determine the closure relation of K_θ-orbits in $N(\mathfrak{p})$ for the classical symmetric pairs. This is completed in §2 by means of a certain ordering of *ab*-diagrams.

For the classical Lie algebras, the nilpotent orbits are classified by Young diagrams, and their closure relation is described by a certain ordering of Young diagrams. Then Kraft and Procesi ([KP2], [KP3]) showed that the smooth equivalence class (cf. §3) $\mathrm{Sing}(\bar{\mathcal{O}}_\eta, \mathcal{O}_\sigma)$ of the closure $\bar{\mathcal{O}}_\eta$ in $\mathcal{O}_\sigma$, which corresponds to a degeneration $\sigma\leq\eta$ of Young diagrams, is determined only by reduced degeneration $\bar{\sigma}\leq\bar{\eta}$, i.e., the degeneration which we obtain from $\sigma\leq\eta$ by erasing the common columns and rows from the left and the upside.

The second purpose of this paper is to give an analogue of the result of Kraft and Procesi for the classical symmetric pairs. The construction $\overline{C_\eta^{(\varepsilon,\omega)}} \xleftarrow{\rho} N_\eta \xrightarrow{\pi} \overline{C_{\eta'}^{(-\varepsilon,-\omega)}}$ (cf. §3), which we need to prove the "cancelling columns", is also used to give a reduction to determine the closure relation.

On the other hand, there exists a natural correspondence between symmetric pairs and real Lie groups. Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair defined by (G, θ) and let $G_{\boldsymbol{R}}$ be the corresponding real group with Lie algebra $\mathfrak{g}_{\boldsymbol{R}}$. Then it is known by Sekiguchi [Se2] that there is a natural correspondence between the set of nilpotent K_θ-orbits in $\mathfrak{p}$ and that of nilpotent $G_{\boldsymbol{R}}$-orbits in $\mathfrak{g}_{\boldsymbol{R}}$. We call this correspondence Sekiguchi's bijection. Then we are naturally led to the problem whether Sekiguchi's bijection preserves the