Tôhoku Math. J.
43 (1991), 425–445

THE REPRESENTATION THEORY OF INNER ALMOST COMPACT FORMS OF KAC-MOODY ALGEBRAS

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(Received June 18, 1990)

1. Introduction. Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure. In [A] we presented a construction by generators and relations of k-forms of (symmetrizable) "derived" Kac-Moody algebras over $\bar{k}$, under certain restrictions. This construction can be roughly described as *glueing together* suitably chosen three-dimensional simple Lie algebras (TDS for short) over k. (Let us recall that the TDS's over k are in one-to-one correspondence with the quaternion algebras over k; hence the notation $sq(a, b)$, see Section 1). It was shown in [AR] that, in the real case, these forms are inner "almost compact", using Rousseau's terminology.

Another approach is followed in [BP]. The classification of the real forms of the first kind of affine Lie algebras is contained in [L], see also [BR].

In this paper we extend the results of [A] and construct forms of (non-derived) Kac-Moody algebras. We drop also here the requirement $a_{ij} \geq -3$ of [A]. As in the quoted paper, we are also able to construct a symmetric bilinear invariant form.

Lie algebras become more interesting when (some of) its representation theory is understood. In the Kac-Moody case, the theory of highest weight modules, inspired by the finite case, has many deep connections with other areas of Mathematics: see for example [K]. Again, the theory relies on the $sl(2)$-case.

In this article, we propose a definition of "quadratic" highest weight modules for the introduced forms. As in the split case, we need first to understand the $sq(a, b)$-case, (cf. Section 4).

Let us emphasize that we have no longer the notion of Borel subalgebra, nor is Lie's theorem applicable, and the action of the Cartan subalgebra is not in general diagonalizable. We can however, manage the situation and define a "quadratic" highest weight module for each non-zero element of the dual of the Cartan subalgebra as a cyclic one, subject to some quadratic relations. In Sections 6 and 7 we extend this definition to the general case.

As a first application, we give a presentation of the "derived" forms of Kac-Moody algebras (in the spirit of Gabber-Kac's theorem) generalyzing formulas (13), $\cdots$, (16), (23), $\cdots$, (26) of [A], (cf. Section 5).

This work was supported by CONICET, CONICOR and FAMAF (Argentina).