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ON THE RATIONAL STRUCTURES OF SYMMETRIC DOMAINS, II DETERMINATION OF RATIONAL POINTS OF CLASSICAL DOMAINS

Dedicated to the memory of Michio Kuga

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In Part I of this series, quoted hereafter as [Sa2], we gave an algebraic formulation of the Siegel domain realization of symmetric domains and applied it to the determination of "rational points" of symmetric domains with $\boldsymbol{Q}$-structure. To be more precise, let $\mathfrak{g}$ be a (real) semisimple Lie algebra of hermitian type defined over $\boldsymbol{Q}$, $\boldsymbol{Q}$-simple and of $\boldsymbol{Q}$-rank r_0. Let $\mathscr{D}$ be a symmetric domain associated with $\mathfrak{g}$, which (as a set) may be identified with the set of Cartan involutions of $\mathfrak{g}$. A point of $\mathscr{D}$ is called *rational* if the corresponding Cartan involution is defined over $\boldsymbol{Q}$. We showed in [Sa2] (Th. 3) that, if $r_0>0$, then the determination of rational points is essentially reduced to that for the "last" (i.e., the r_0-th) rational boundary components, which are always of classical type. By virtue of the isomorphisms between classical groups, it is known that all classical domains with $r_0=0$ are realized as a domain of type (I) (see §1 of this paper). The main purpose of this Part II is to give an actual determination of rational points in the case of domains of type (I).

The semisimple Lie algebra $\mathfrak{g}$, or the associated symmetric domain $\mathscr{D}$, is called *pure* if all $\boldsymbol{R}$-simple factors of $\mathfrak{g}$ are $\boldsymbol{R}$-isomorphic to one another. It is called *strictly pure* if all $\boldsymbol{Q}$-simple factors in the reductive part of the $\boldsymbol{Q}$-parabolic subalgebras corresponding to rational boundary components of $\mathscr{D}$ are pure. (Note that these two conditions are actually equivalent except for the case where $\mathfrak{g}$ is of type (D_l^m).) The results in [Sa2] (Lem. 3, Th. 3) imply that, if $\mathscr{D}$ has rational points, then $\mathfrak{g}$ is strictly pure. For the domains of tube type the converse of this is also true except for the case of domains of type (I), which is discussed in detail in this paper. A part of our results was obtained by K. Oiso in his Master thesis. It is given here in a refined form with a different proof.

Sections 1 and 2 are mostly of preliminary nature. In §1 we give a list of pure $\boldsymbol{Q}$-simple classical Lie algebras of hermitian type and in §2 summarize some basic facts on "unitary involutions" (i.e., involutions of the second kind) and hermitian forms. Then in §3 we explain a method to determine rational points of a domain $\mathscr{D}$ of type $(\mathrm{I}_{p,q})^m$ and give a necessary condition for the existence of rational points (Th. 4). The necessary and sufficient conditions for the existence of rational points (with a given CM-field) are given in §4 (Th. 5, 6, 7). Using these results, we discuss in §5 the rational