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LINEAR GALE TRANSFORMS AND GELFAND-KAPRANOV-ZELEVINSKIJ DECOMPOSITIONS

Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

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Abstract. In the convex-geometric setting of what we call linear Gale transforms and convex polyhedral cone decompositions, we generalize and reformulate results on (1) the secondary polytope of a convex polytope considered by Gelfand, Kapranov and Zelevinskij in connection with the discriminants of projective toric varieties, as well as (2) the wall geometry of fans considered by Reid in connection with Mori's birational geometry in the particular case of projective toric varieties.

Introduction. Let Ξ be a finite subset of an r-dimensional vector space W over the field $\boldsymbol{R}$ of real numbers. As one of us already sketched in [14, §2.6], let us now outline our results in this paper *assuming, for simplicity, that* Ξ *spans* W *over* $\boldsymbol{R}$, *that* Ξ *does not contain* 0 *and that each* $\xi \in \Xi$ *is not a positive scalar multiple of any other element in* $\Xi \setminus \{\xi\}$.

Among the pairs (V, f) of an $\boldsymbol{R}$-vector space V and a map $f: \Xi \to V$ such that $f(\Xi)$ spans V over $\boldsymbol{R}$ and that

$$\sum_{\xi \in \Xi} \xi \otimes f(\xi) = 0 \qquad \text{in} \quad W \otimes_{\boldsymbol{R}} V,$$

there exists a pair $(G(W, \Xi), g)$, called the *linear Gale transform* of (W, Ξ), satisfying the following *universality*: For each (V, f) there exists a unique $\boldsymbol{R}$-linear map h: $G(W, \Xi) \to V$ such that $f = h \circ g$.

Convex-geometric and combinatorial properties for Ξ turn out to be reflected in those for $g(\Xi)$ in an interesting way. See Shephard [16] and McMullen [11], where the appellations *linear representation* and *linear transform* are used. Consider, for instance, convex polyhedral cones

$$W_{\geq 0}(\Xi) := \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} \xi \quad \text{and} \quad G_{\geq 0}(W, \Xi) := \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} g(\xi)$$

in W and $G(W, \Xi)$, respectively, spanned by Ξ and $g(\Xi)$ over the additive semigroup

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