Tôhoku Math. J.
43 (1991), 361–374

CURVATURE PINCHING THEOREM FOR MINIMAL SURFACES WITH CONSTANT KAEHLER ANGLE IN COMPLEX PROJECTIVE SPACES

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(Received May 29, 1990, revised January 10, 1991)

Introduction. Let X be a complex space form with the complex structure J and the standard Kaehler metric $\langle\,,\,\rangle$, M be an oriented 2-dimensional Riemannian manifold and $x: M \to X$ be an isometric minimal immersion of M into X. Then the Kaehler angle α of x, which is an invariant of the immersion x related to J, is defined by $\cos(\alpha) = \langle Je_1, e_2\rangle$, where $\{e_1, e_2\}$ is an orthonormal basis of M. The Kaehler angle gives a measure of the failure of x to be a holomorphic map. Indeed x is holomorphic if and only if $\alpha = 0$ on M, while x is anti-holomorphic if and only if $\alpha = \pi$ on M. In [4], Chern and Wolfson pointed out that the Kaehler angle of x plays an important role in the study of minimal surfaces in X. From this point of view, we would like to know all isometric minimal immersions of constant Kaehler angle in X.

In this paper, we shall mainly discuss this problem when X is a complex space form of positive constant holomorphic sectional curvature. So, let $P^n(C)$ be the complex projective space with the Fubini-Study metric of constnat holomorphic sectional curvature 4ρ. Let $S^2(K)$ be a 2-dimensional sphere of constant Gaussian curvature K. Examples of minimal surfaces of constant Kaehler angle in $P^n(C)$, are given in [1] and [2]: For each integer p with $0 \le p \le n$, there exist full isometric minimal immersions $\varphi_{n,p}: S^2(K_{n,p}) \to P^n(C)$, where $K_{n,p} = 4\rho/(n + 2p(n-p))$. Each $\varphi_{n,p}$ possesses holomorphic rigidity, that is to say, such two immersions differ by a holomorphic isometry of $P^n(C)$. The Kaehler angle $\alpha_{n,p}$ of $\varphi_{n,p}$ is given by $\cos(\alpha_{n,p}) = (n-2p)/(n+2p(n-p))$. Note that $K_{n,p} = 2\rho(1-(2p+1)\cos(\alpha_{n,p}))/p(p+1)$.

Characterizing minimal surfaces of constant Kaehler angle in $P^n(C)$, Ohnita [10] recently gave the following theorem: Let $\varphi: M \to P^n(C)$ be a full isometric minimal immersion of a 2-dimensional Riemannian manifold M into $P^n(C)$. Assume that the Gaussian curvature K of M and the Kaehler angle α of φ are both constant on M. Then the following hold.

(1) If $K > 0$, then there exists some p with $0 \le p \le n$ such that $K = 4\rho/(n + 2p(n-p))$, $\cos(\alpha) = (n-2p)/(n+2p(n-p))$ and $\varphi(M)$ is an open submanifold of $\varphi_{n,p}(S^2(K))$.

(2) If $K = 0$, then $\cos(\alpha) = 0$, that is to say, φ is totally real. Such φ's were already classified by Kenmotsu [6].

(3) The case of $K < 0$ is impossible.

In [10], Ohnita conjectured that the theorem will hold without the assumption