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ON THE CLASSIFICATION OF SMOOTH PROJECTIVE TORIC VARIETIES

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Abstract. We investigate the problem of the classification of smooth projective toric varieties V of dimension d with a given Picard number ρ over an algebraically closed field. For that purpose we introduce a convenient combinatorial description of such varieties by means of primitive relations among $d+\rho$ integral generators of the associated complete regular fan of convex cones in d-dimensional real space. The main conjecture asserts that the number of the primitive relations is bounded by an absolute constant depending only on ρ. We prove this conjecture for $\rho \leq 3$ and give the classification of d-dimensional smooth complete toric varieties with $\rho = 3$.

1. Introduction. Let k be an arbitrary algebraically closed field. A d-dimensional algebraic torus T is a product of d copies of the multiplicative group k^* of k. A toric variety V is a normal algebraic variety containing T as a Zariski open dense subset with an algebraic action of T on V which extends the group law of T. Any toric variety can be described by a finite system of cones spanned by integer points in the real space $\boldsymbol{R}^d$. The reader is referred to [1] for the precise definitions.

In this paper we restrict ourselves to complete smooth toric varieties V. Moreover, we shall often assume that V is a projective toric variety.

One can notice that any description of smooth toric varieties has two sides: the combinatorial structure of the corresponding fan and unimodularity conditions on its generators. The weighted triangulations of $(d-1)$-dimensional sphere introduced in [7] is an example of such a description. One of our objectives is to give a new description of complete smooth toric varieties.

In §2 we introduce the notion of a *primitive collection* of generators and the notion of an associated *primitive relations* among generators. We use these notions to describe toric varieties. If a toric variety V is projective we define also the *degree* of a primitive relation and the *distance* between a generator and a d-dimensional cone of the corresponding fan $\Sigma(V)$.

All these notions are used in §3 to get some properties of the combinatorial structure of a d-dimensional fan $\Sigma(V)$ associated with a toric variety V. It should be remarked that if the Picard number $\rho(V) \geq 3$ there exist combinatorial types of simplicial polytopes which do not give rise to any complete regular fan defining a smooth toric variety [2]. We prove that an arbitrary d-dimensional projective regular fan of cones has a primitive

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