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A TRANSPLANTATION THEOREM FOR LAGUERRE SERIES

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1. Introduction. Let $L_n^\alpha(x)$, $\alpha > -1$, be the Laguerre polynomial of degree n and of order α defined by

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!}\left(\frac{d}{dx}\right)^n (e^{-x}x^{n+\alpha}) .$$

Then the functions $\tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2}$, $n=0, 1, 2, \cdots$, are orthonormal on the interval $(0, \infty)$ with respect to the ordinary Lebesgue measure dx, where

$$(\tau_n^\alpha)^{-2} = \int_0^\infty \{L_n^\alpha(x)\}^2 e^{-x} x^\alpha \, dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} .$$

This orthonormal system leads us to the formal expansion of a function $f(x)$ on $(0, \infty)$:

$$f(x) \sim \sum_{n=0}^\infty a_n^\alpha(f) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2} ,$$

where $a_n^\alpha(f)$ is the n-th Fourier-Laguerre coefficient of order α of $f(x)$ defined by

$$a_n^\alpha(f) = \int_0^\infty f(x) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2} \, dx .$$

We note that the integral converges and $a_n^\alpha(f)$ is finite if $\alpha \geq 0$ and $1 \leq p \leq \infty$, or if $-1 < \alpha < 0$ and $(1+\alpha/2)^{-1} < p \leq \infty$.

Our theorem is as follows:

THEOREM. *Let $\alpha, \beta > -1$ and $\gamma = \min\{\alpha, \beta\}$. If $\gamma \geq 0$, then*

$$\int_0^\infty \left| \sum_{n=0}^\infty a_n^\beta(f) \tau_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2} \right|^p dx \leq C \int_0^\infty |f(x)|^p \, dx \tag{1.1}$$

for $1 < p < \infty$, where C is a constant independent of f. If $-1 < \gamma < 0$, then (1.1) *holds for* $(1+\gamma/2)^{-1} < p < -2/\gamma$.

Historically, Guy [11] proved a transplantation theorem for Hankel transforms. Schindler [14] proved Guy's theorem showing an explicit integral representation. For other classical expansions, Askey and Wainger [3], [4] gave transplantation theorems for ultraspherical coefficients and its dual. Furthermore, Askey [1], [2]