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GORENSTEIN TORIC SINGULARITIES AND CONVEX POLYTOPES

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Introduction. Let X be a normal projective variety over a field k and D an *ample $\boldsymbol{Q}$-divisor*, i.e., a rational coefficient Weil divisor such that bD is an ample Cartier divisor for some positive integer b. We consider a normal graded ring $R(X, D)$ defined by $R(X, D)=\bigoplus_{n\geq 0}H^0(X, \mathcal{O}_X(nD))t^n$. Here t is an indeterminate and $\mathcal{O}_X(nD)$ are the sheaves defined by $\Gamma(U, \mathcal{O}_X(nD))=\{f\in K(X);\ \mathrm{div}_U(f)+nD|_U\geq 0\}$ for each open set U of X. We are interested in finding a criterion for a normal projective variety X to have an ample $\boldsymbol{Q}$-divisor D with $R(X, D)$ Gorenstein. Concerning this problem, see also [1, Chapter 5], [10]. Here we discuss this problem when X is a projective torus embedding defined over k.

Our main results are the following:

COROLLARY 2.5. *Let X be a projective torus embedding and D an ample Cartier divisor. Then $R(X, D)$ is Gorenstein if and only if the canonical sheaf ω_X on X is isomorphic to an invertible sheaf $\mathcal{O}_X(-\delta D)$ for a positive integer δ.*

THEOREM 2.6. *Every projective torus embedding X has an ample $\boldsymbol{Q}$-divisor D stable under the torus action such that $R(X, D)$ is Gorenstein.*

To obtain these results, we proceed as follows: First, given a *rational convex r-polytope P* in $\boldsymbol{R}^r$ (i.e., an r-dimensional convex polytope whose vertices have rational coordinates in $\boldsymbol{R}^r$), we construct a pair of projective torus embedding $X(P)$ over k and an ample $\boldsymbol{Q}$-divisor $D(P)$ (Proposition 1.3) following [7, Chapter 2], so that $R(X(P), D(P))$ is isomorphic to the normal semigroup k-algebra

$$R(P)=\bigoplus_{n\geq 0}\left\{\sum_{m\in nP\cap \boldsymbol{Z}^r} k\boldsymbol{e}(m)\right\}t^n .$$

Here t is an indeterminate and $\boldsymbol{e}$ is the isomorphism from $\boldsymbol{Z}^r$ ($\subset \boldsymbol{R}^r$) into the Laurent polynomial ring $k[X_1^{\pm 1}, \dots, X_r^{\pm 1}]$ sending $(m_1, \dots, m_r)$ to $X_1^{m_1}\cdots X_r^{m_r}$. Thus $X(P)$ is isomorphic to $\mathrm{Proj}(R(P))$ (Proposition 1.5). Conversely, it turns out that every pair of projective torus embedding X and a T-stable ample $\boldsymbol{Q}$-divisor D on X is obtained from a rational convex r-polytope in $\boldsymbol{R}^r$ in this way (Proposition 1.3). On the other hand, since $R(X(P), D(P))$ ($\simeq R(P)$) is Cohen-Macaulay (cf. [4]), we can apply the criterion [10, Corollary 2.9] for the Gorenstein property to $R(X(P), D(P))$. Therefore we have Proposition 2.2, which is a criterion for $R(X(P), D(P))\simeq R(P)$ to be Gorenstein in terms