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ON NORMAL AND CONORMAL MAPS FOR AFFINE HYPERSURFACES

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Abstract. We prove two main results in affine differential geometry that characterize ellipsoids among the ovaloids. The first theorem states that an ovaloid in the 3-dimensional affine space is an ellipsoid if and only if the Laplacian of the normal map is proportional to the normal map. The second theorem says that a hyperovaloid in an affine space of any dimension is a hyperellipsoid if and only if the conormal image (or the normal image) is a hyperellipsoid with center at the origin.

Let $f: M^n \to R^{n+1}$ be a nondegenerate hypersurface with affine normal ξ in the affine space R^{n+1}. We then have the normal map $\phi: M^n \to R^{n+1}$ and the conormal immersion $v: M^n \to R_{n+1}$, where R_{n+1} is the coaffine space of R^{n+1} (for the terminology see [N-P]). Our main results are the following.

THEOREM 1. *An ovaloid $f: M^2 \to R^3$ is an ellipsoid if and only if the Laplacian of the normal map $\phi: M^2 \to R^3$ is proportional to ϕ.*

THEOREM 2. *For a hyperovaloid $f: M^n \to R^{n+1}$, $n \geq 2$, the following three conditions are equivalent*:
(1) *The conormal image $v(M^n)$ is a hyperellipsoid with center at the origin of R_{n+1}.*
(2) *The normal image $\phi(M^n)$ is a hyperellipsoid with center at the origin of R^{n+1}.*
(3) *$f(M^n)$ is a hyperellipsoid.*

In Section 1 we study the normal and conormal maps for nondegenerate hypersurfaces. By using the notion of conjugate connection we express the relationships between the three connections that arise when the normal map is an immersion. In Section 2 we compute the Laplacian of the normal map and prove Theorem 1. We may prove Theorem 2 in the case $n=2$ using the same method, but the general case of Theorem 2 requires a different approach and this is presented in Section 3.

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