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INTEGRO-DIFFERENTIAL EQUATIONS AND DELAY INTEGRAL INEQUALITIES

XU DAOYI[1]

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Abstract. In this paper sufficient conditions for the boundedness, asymptotic properties and exponential decay are first obtained for solutions of linear systems of integral inequalities with infinite delay. Then nonlinear integro-differential equations are reduced to delay integral inequalities by the variation of parameter formula, and some criteria are given for asymptotic stability, uniformly asymptotic stability and exponential asymptotic stability. The results obtained here are illustrated by examples which have been particularly difficult to treat by means of the standard Lyapunov theory.

1. Introduction. This paper is concerned with asymptotic behavior and stability of solutions of the integro-differential equation

$$\dot{x}(t)=A(t)x(t)+f[t, x(r_1(t))]+\int_{\alpha}^{t} G[t, s, x(r_2(s))]ds\,, \tag{1}$$

where $A(t)$ is a continuous $n\times n$ matrix on $[0, \infty)$, $r(t)\leq r_1(t)$, $r_2(t)\leq t$ and $r(t)\to\infty$ as $t\to\infty$.

In this discussion, R^n denotes the n-dimensional Euclidean space, $R^+=[0, \infty)$ and $C[X, Y]$ the class of continuous mappings from the topological space X to the topological space Y. $C=C([\alpha, 0], R^n)$, in which $\alpha\leq t$ could be $-\infty$. For $\phi\in C$ we define $\|\phi\|_\alpha=\sup_{\alpha\leq u\leq t}|\phi(u)|$, where $|\cdot|$ is a norm in R^n.

It is assumed that $f\in C[R^+\times C, R^n]$ and $G\in C[R^+\times R\times C, R^n]$. For any $t_0\geq 0$ and any $\phi\in C$, a solution of (1) is a function $x: R\to R^n$ satisfying (1) for $t\geq t_0$ and that $x(t)=\phi(t)$ for $-\infty<t\leq t_0$. Throughout this paper we always assume that (1) has a continuous solution denoted by $x(t, t_0, \phi)$ or simply $x(t)$ if no confusion should arise.

We refer the reader to [1] or [6] for the definitions of the terms we use on stability. We always assume that $f(t, 0)\equiv G(t, s, 0)\equiv 0$ in our discussion of stability.

If $r(t)\equiv t$ in (1), then (1) becomes a familiar integro-differential equation investigated extensively by a number of authors (see Burton [1], Hara, Yoneyama and Itoh [6], Kato [8] and Murakami [10] and their bibliographies). To avoid difficulty in constructing the Lyapunov functional, Gopalsamy [5] dealt with the systems of the type (1) with $r_1(t)=t-r$ (r is a constant) and $r_2(t)=t$ using the inequality technique, while Hara, Yoneyama and Itoh [6] dealt with the case with $r_2(t)=t$ and $f\equiv 0$ using the "variation of parameters" formula. Some "easily verifiable" sufficient conditions

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