Tôhoku Math. J.
45 (1993), 393–422

HOLOMORPHIC MAPS FROM COMPACT MANIFOLDS INTO LOOP GROUPS AS BLASCHKE PRODUCTS

GIORGIO VALLI

(Received March 2, 1992, revised November 19, 1992)

Abstract. We describe a factorization theorem for holomorphic maps from a compact manifold M into the loop group of $U(N)$. We prove that any such map is a finite Blaschke product of maps into Grassmann manifolds (unitons), satisfying recursive holomorphicity conditions; each map being attached to a point in the open unit disc. This factorization is essentially unique. Using a theorem of Atiyah and Donaldson, we construct a stratification of the moduli space of framed $SU(2)$ Yang-Mills instanton over the 4-sphere, in which the strata are iterated fibrations of spaces of polynomials, indexed by plane partitions; and the unique open stratum of "generic" instantons of charge d, is the configuration space of d distinct points in the disc, labelled with d biholomorphisms of the 2-sphere.

Introduction. Let $\Omega U(N)=\{\gamma: S^1 \to U(N) \mid \gamma \text{ real analytic}, \gamma(1)=I\}$ be the real analytic loop group of the unitary group $U(N)$. By using Fourier series expansions, $\Omega U(N)$ may be given a Kähler manifold structure (cf. [A]).

In this paper we study holomorphic maps (or, more generally, rational maps (cf. the definition in §2), from a compact complex manifold M into $\Omega U(N)$.

The motivation for this study comes from two different results, both in the realm of gauge theory and twistor geometry.

(1) By a theorem of Atiyah and Donaldson (cf. [A]), for any classical group G, the parameter space of based holomorphic maps $S^2 \to \Omega G$ is diffeomorphic to the space of Yang-Mills instantons over S^4, modulo based gauge transformations. The instanton number corresponds to the degree of the map, defined via $H^2(\Omega G, \boldsymbol{Z}) \cong \boldsymbol{Z}$.

(2) Uhlenbeck [U] associated a holomorphic map $F: S^2 \to \Omega U(N)$ to any harmonic map $f: S^2 \to U(N)$, using methods from the theory of completely integrable systems. She gave a recursive procedure, similar to a Bäcklund transformation, to generate new F's from given ones by the choice of appropriate holomorphic vector bundles over S^2, called unitons. Then she proved a unique factorization theorem of any such F as a product of unitons.

Moreover, generalizing the paper of Uhlenbeck, Segal [Seg] has showed that any holomorphic map from a compact manifold into $\Omega U(N)$ has values in the space of rational loops. But it is relatively well known that any based rational matrix valued function, unitary on the circle, has a finite factorization as a "Blaschke product" (cf. [G]).

Key words. Loop group, Blaschke product, instanton, uniton.
1991 *Mathematics Subject Classification.* Primary 81T13; Secondary 14F05, 32H02, 32L25, 47A56.