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LIAPUNOV FUNCTIONALS AND PERIODICITY IN INTEGRAL EQUATIONS

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Abstract. Liapunov's direct method has been used very effectively for a hundred years on various types of differential equations. It has not, however, been used with much success on non-differentiated equations. In this paper we construct a Liapunov function for a nonlinear integral equation with an infinite delay which is nonconvolution type. From that Liapunov function we deduce conditions for boundedness, stability, and the existence of periodic solutions. The kernel of the integral equation is a perturbation of a positive kernel and there are estimates showing how large the perturbation can be. The advantage of the Liapunov approach over classical methods for integral equations is the simplicity of analysis, once a Liapunov function is constructed.

1. Introduction. Liapunov functions and functionals have been used very effectively on ordinary, functional, and partial differential equations, but have had little application to nondifferentiated equations (cf. Miller [12; p. 337] and Gripenberg et al. [5; p. 426]). The reason for this is simple. Given

$$x'=f(t,x)\,,\quad '=d/dt\,,$$

and any differentiable scalar function

$$V(t,x)\,,$$

if $x(t)$ is a solution, then $V(t,x(t))$ is a scalar function of t and we can compute

$$dV(t,x(t))/dt=\operatorname{grad} V\cdot f+\partial V/\partial t\,.$$

That is, we can find the derivative of V along the solution directly from the differential equation. If it turns out, for example, that $dV/dt\le 0$, then this may yield much information about the behavior of the unknown solution.

By contrast, if

$$x(t)=a(t)+\int_{-\infty}^{t} g(t,s,x(s))ds\,,$$

it seems unclear how to relate the derivative of a scalar function $V(t,x)$ to the unknown solution. Indeed, Miller [12; p. 337] proceeds only under the assumption that the inte-

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