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THE SPECTRAL TYPE OF THE STAIRCASE TRANSFORMATION

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Abstract. We show that a certain Riesz-product type measure is singular. This proves the singularity of the spectral measures of a certain ergodic transformation, known as the staircase.

Introduction. The staircase transformation is an example of a "rank one" transformation whose properties have been of interest in ergodic theory recently [Adams], [Adams, Friedman], [Choksi, Nadkarni]. Here we prove that it has singular spectrum. To be more precise, this means that the maximal spectral type of the induced unitary operator is singular with respect to the Lebesgue measure on the circle. We refer the reader to [Choksi, Nadkarni, (example 2)] for the definition of the staircase transformation and for other background information. It is shown there that the problem reduces to proving the singularity of a specific measure μ, which is defined as follows. Let h_n, $n=1, 2, \cdots$ be the integers defined inductively by

$$h_1=1\,,\quad h_{n+1}=nh_n+(1+2+3+\cdots+n)\,.$$

Define trigonometric polynomials $P_n(z)$, where $z=e^{i\theta}$, $\theta\in[0, 2\pi)$, by

$$P_n(z)=\frac{1}{\sqrt{n}}(1+z^{h_n+1}+z^{2h_n+1+2}+z^{3h_n+1+2+3}+\cdots+z^{(n-1)h_n+n(n-1)/2})\,.$$

If λ denotes the normalized Lebesgue measure on $[0, 2\pi)$ the measures $\prod_{n=1}^{N}|P_n|^2d\lambda$ turn out to have weak* limit $d\mu$. The purpose of this paper is to prove that $\mu\perp\lambda$.

THEOREM. $\mu\perp\lambda$.

For this theorem, the reader will not need to know the ergodic theory background. Only the definitions of the polynomials P_n are really used. The overall method of the proof is based on [Bourgain]. Then some specific properties of the above polynomials are needed to make the method work in this case.

The proof actually gives more than the statement of the theorem. It gives the same result for other "staircase constructions". By this we mean that one can have polynomials P_{n_j}, of the above type, but with h_n replaced by h_j where $h_{j+1}=n_jh_j+(1+2+\cdots+n_j)$.

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