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ON CONTRACTIBLE CURVES IN THE COMPLEX AFFINE PLANE

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Abstract. Concerning the topologically contractible curves embedded in the affine plane defined over the complex numbers we shall present new conceptual proofs to the theorem of Abhyankar-Moh and the theorem of Lin-Zaidenberg which are based on the structure theorems of non-complete algebraic surfaces.

1. Introduction. In this paper we give new proofs of the following two results from affine algebraic geometry.

THEOREM 1. *Let $C \subset \boldsymbol{C}^2$ be a closed embedding of the affine line A^1. Then there is an algebraic automorphism of $\boldsymbol{C}^2$ which maps C onto the line $\{X=0\}$, where X, Y are some affine coordinates on $\boldsymbol{C}^2$.*

THEOREM 2. *Let $C \subset \boldsymbol{C}^2$ be an irreducible algebraic curve which is topologically contractible. Then there exist affine coordinates X, Y on $\boldsymbol{C}^2$ such that in terms of these coordinates C is defined by the equation $\{X^m = Y^n\}$, where* $\gcd(m, n) = 1$.

Theorem 1 was first proved by Abhyankar and Moh [1] and independently by Suzuki [21]. Later on, several proofs of this result were found chronologically by Miyanishi [14], Rudolph [20], Richman [19] and Kang [8].

Most of these proofs involve either somewhat heavy calculations or detailed analysis of the singularity at infinity for the curve C. The proof of Rudolph is short and based on some basic results from knot theory (hence somewhat inaccessible to algebraic geometers).

Theorem 2 was first proved by Lin and Zaidenberg [13]. Their proof involves deep results from Teichmüller theory.

In view of the importance of these results for affine algebraic geometry, it is useful to have different ways of looking at these results. Our proofs of these results use essentially the same ideas from the theory of non-complete algebraic surfaces. An inequality of Miyaoka-Yau type proved by R. Kobayashi, S. Nakamura and F. Sakai plays a crucial role in the proofs. The results from the theory of non-complete algebraic varieties that we use have become by now well-known (except possibly for the inequality of Miyaoka-Yau type). Once the basic results about non-complete algebraic surfaces are assumed, our proofs become rather short and, we believe, more conceptual. In

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