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PERIODIC SOLUTIONS OF DISSIPATIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. We consider periodic, infinite delay differential equations. We investigate dissipativeness for these equations. Massat proved that dissipative, periodic, infinite delay equations have a periodic solution. For our purpose we need a weaker dissipativeness, so we prove Massat's theorem from this weak dissipativeness in an elementary way. Then we extend a theorem of Pliss giving a necessary and sufficient condition for this weak dissipativeness. We also present a theorem using Liapunov functionals to show the weak dissipativeness and hence the existence of a periodic solution.

1. Introduction. Let $f: \boldsymbol{R}\times\boldsymbol{R}^d\to\boldsymbol{R}^d$ be continuous and locally Lipschitz in x with $f(t+T, x)=f(t, x)$ for all (t, x) and some $T>0$. We say that the ordinary differential equation

$$x'=f(t, x) \tag{1}$$

is dissipative, if all solutions become bounded by a fixed constant at some time and remain bounded from that time on. Pliss [9, Theorem 2.1] showed that the ordinary differential equation is dissipative if and only if there is an $r>0$ such that for each (t_0, x_0) there is a $\tau>t_0$ with $|x(\tau, t_0, x_0)|<r$. The author [7] generalized this result for finite delay differential equations stating that dissipativeness is equivalent to every solution becoming bounded by a fixed constant for an interval of length $2h$, where h is the retardation. The author also gave an elementary proof for a result of Hale and Lopes [3], who proved that dissipativeness implies the existence of a periodic solution for finite delay equations. The following Lyapunov-type theorem, which can also be found in [7], proves the existence of a periodic solution through dissipativeness.

THEOREM A. *Suppose there are a functional $V: \boldsymbol{R}\times\mathscr{C}\to\boldsymbol{R}$ and constants $a, b, M, U>0$ such that*

(i) $0\le V(t, \phi)$,

(ii) $V'(t, x_t)\le M$ *and*

(iii) $V'(t, x_t)\le -a|x'(t)|-b$ *for* $|x(t)|\ge U$.

Then the solutions of the finite delay differential equation are dissipative.

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