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OSCILLATOR AND PENDULUM EQUATION ON PSEUDO-RIEMANNIAN SPACES

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Abstract. We study conformal vector fields on pseudo-Riemannian manifolds which are locally gradient fields. This is closely related with a certain differential equation for the Hessian of a real function. We obtain global solutions of the oscillator and peandulum equation for the Hessian of this function on a pseudo-Riemannian manifold, generalizing previous results by M. Obata, Y. Tashiro, and Y. Kerbrat. In particular, it turns out that the pendulum equation characterizes a certain conformal type of metrics carrying a conformal vector field with infinitely many zeros.

1. Introduction. Conformal mappings and conformal vector fields are classical topics in geometry. Essential conformal vector fields on Riemannian spaces were studied by Obata, Lelong-Ferrand and Alekseevskii [A1], [La2]. Conformal gradient fields are essentially solutions of the differential equation $\nabla^2\varphi=(\Delta\varphi/n)\cdot g$. This equation was studied since the 1920's by Brinkmann, Fialkow, Yano, Tashiro, Kerbrat and others. In the Riemannian case the results are quite complete. In the pseudo-Riemannian case a systematic approach has started in our previous paper [KR2] including a conformal classification theorem.

A classical result by Obata and Tashiro characterizes the standard sphere as the only complete Riemannian manifold admitting a non-constant solution of the equation $\nabla^2\varphi=-c^2\varphi g$ for a non-zero constant c. This is nothing but the classical *harmonic oscillator equation*. In Section 3 we study the following generalization: given a function $h\colon \boldsymbol{R}\to\boldsymbol{R}$, the conformal gradient field equation $\nabla^2\varphi+h(\varphi)\cdot g=0$ imposes very strong conditions on the underlying pseudo-Riemannian manifold. We give analogous results for the case of the equation of the general *undamped oscillator*. This illustrates how a metric can be modeled within a conformal class by a second order differential equation. The metric in this case is completely determined by the equation and the choice of a constant of integration which is essentially the energy of the undamped oscillator. Similarly, for small energy, the pendulum equation on a pseudo-Riemannian manifold determines the metric uniquely. In the Riemannian case it is conformal to the standard sphere whereas in the case of an indefinite metric it is conformal to a noncompact manifold $M(\boldsymbol{Z})$. This manifold carries a conformal gradient field with infinitely many zeros. A short announcement of the results in this paper appeared in [KR3].

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