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ON THE INTERIOR SPIKE LAYER SOLUTIONS TO A SINGULARLY PERTURBED NEUMANN PROBLEM

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Abstract. In this paper, we construct interior spike layer solutions for a class of semilinear elliptic Neumann problems which arise as stationary solutions of Keller-Segel model in chemotaxis and also as limiting equations for the Gierer-Meinhardt system in biological pattern formation. We also classify the locations of single interior peaks. We show exactly how the geometry of the domain affects the spike solutions.

1. Introduction.

Consider the problem

$$
\begin{cases} \varepsilon^2\Delta u-u+u^p=0 & \text{in } \Omega\,, \\ u>0 & \text{in } \Omega\,, \\ \partial u/\partial\nu=0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}
$$

where $\Omega\subset R^N$, $\varepsilon>0$, $1<p<(N+2)/(N-2)$ when $N\geq 3$, and $1<p<\infty$ when $N=1, 2$ and ν is the outward normal vector to $\partial\Omega$.

Equation (1.1) is known as the stationary equation of the Keller-Segel system in chemotaxis. It can also be seen as the limiting stationary equation of the so-called Gierer-Meinhardt system in biological pattern formation. (See [11] for more details.)

In the pioneering papers of [7], [9] and [10], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for ε sufficiently small the least-energy solution has only one local maximum point P_ε and $P_\varepsilon\in\partial\Omega$. Moreover, $H(P_\varepsilon)\to\max_{P\in\partial\Omega}H(P)$ as $\varepsilon\to 0$, where $H(P)$ is the mean curvature of $\partial\Omega$ at P. Ni and Takagi [11] constructed boundary spike solutions for axially symmetric domains while in [21], the author studied the general domain case. When $p=(N+2)/(N-2)$, similar results for the boundary spike layer solutions have been obtained by [1], [2], [3], [8], [18], etc.

In all the above papers, only *boundary* spike layer solutions are obtained and studied. It remains to see whether or not *interior* spike layer solutions exist for the problem (1.1). In this paper, we shall study this question and give an affirmative answer.

To state our results, we need to introduce some notation.

By the results of [5] and [6], we know that the solution of the problem

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